Geometric Probability
An ADE Mathematics Lesson
Days 16-20

Author	ADE Content Specialists
Grade Level	10th grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 2: Numerical Operations
PO 3. Calculate powers and roots of rational and irrational numbers.
Concept 3: Estimation
PO 1. Determine rational approximations of irrational numbers.

Strand 2: Data Analysis, Probability, and Discrete Mathematics
Concept 2: Probability
PO 1. Make predictions and solve problems based on theoretical probability models.
PO 5. Use concepts and formulas of area to calculate geometric probabilities.

Strand 4: Geometry and Measurement
Concept 1: Geometric Shapes
PO 1. Use the basic properties of a circle (relationships between angles, radii, intercepted arcs, chords, tangents, and secants) to prove basic theorems and solve problems.
Concept 4: Measurement
PO 2. Find the length of a circular arc; find the area of a sector of a circle.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).
PO 4. Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning.
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.

Connects To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 2. Solve formulas for specified variables.

Strand 4: Geometry and Measurement
Concept 4: Measurement
PO 3. Determine the effect that changing dimensions has on the perimeter, area, or volume of a figure.
PO 4. Solve problems involving similar figures using ratios and proportions.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Overview
It is sometimes necessary to find the probability of an event that involves geometric figures. This lesson will demonstrate how to find geometric probability given various shapes and figures.

Purpose
The purpose of this lesson is to give you more experience working with probability in different problems.

Materials
- Geometric Probability Worksheets
- Ruler

Objectives
Students will:
- Make predictions and solve problems based on theoretical probability models.
- Use concepts and formulas of area to calculate geometric probabilities.

Lesson Components

Prerequisite Skills: In 7th and 8th grades you learned to calculate areas of various figures including composite figures. You learned how to calculate probability given dependent and independent events.

Vocabulary: *probability, geometric probability, dependent event, independent event, expected outcome, composite figure, area*

Session 1 (2 days)

1. Review simple probability.

Session 2 (3 days)

1. Use area formulas and various concepts to calculate geometric probabilities.

Assessment
There is one assessment that follows Session 2 and assesses knowledge of geometric probabilities.

Geometric Probability
Session 1 - Review of Probability

Before working with geometric probability concepts, let's do a short review of some basic probability. It is necessary to understand the terms that follow to be able to work with geometric probability. If you are having difficulty understanding any of them, please review basic probability. Some examples follow the definitions.

- **Theoretical probability** is the likelihood an event will occur under ideal circumstances divided by the total possible outcomes.
- An **outcome** is a possible result for a probability experiment or simulation.
- A **sample space** is a list of all possible outcomes of an activity.
- **Experimental (empirical) probability** is a ratio formed by the comparison of the number of times an event occurs in an experiment to the number of times the experiment is completed.
- **Complementary events** are two events whose probabilities of occurring sum to one; mutually exclusive events (e.g., when flipping a coin, getting a head and getting a tail are complementary events).
- **The Law of Large Numbers** states that the larger the sample the closer the experimental probability will approximate the theoretical probability.
- **Independent events** are two events in which the outcome of the second event is not affected by the outcome of the first event.
- **Dependent events** are two events such that the likelihood of the outcome of the second event is affected by the outcome of the first event.

Theoretical probability is generally calculated before an experiment is conducted. Once the experiment has been completed, the experimental probability can be determined. That experimental probability will not alter the theoretical probability. The larger the sample is, the more likely that the experimental probability will approach the theoretical probability which is called the Law of Large Numbers.

Example 1:

What is the probability that a wooden cube, numbered 1 to 6 on each of its faces, will be a 6 when a person tosses that cube?

Solution:

List the possible outcomes when a wooden cube containing the numbers one to 6 is tossed. There are only 6, which are the following:

$$\{1, 2, 3, 4, 5, 6\}$$

Theoretical probability tells us that the probability of an event is the ratio of the possible number of correct outcomes to the total number of outcomes. In this case, there is one correct outcome (6) over six (1, 2, 3, 4, 5, 6) possible outcomes.

Our answer is 1:6 or can also be written as $\frac{1}{6}$.

Example 2:

What is the probability that the sum of two wooden cubes, each numbered 1 to 6 on their faces, will be a 7 when two cubes are tossed?

Solution:

List the possible outcomes when two wooden cubes containing the numbers 1 to 6 are tossed. There are only 36 outcomes, which are shown in the table; 6 of the outcomes show a sum of 7.

Sum	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Theoretical probability tells us that the probability of an event is the ratio of the possible number of correct outcomes to the total number of outcomes. In this case, there are six correct outcomes (6) over 36 possible outcomes.

Our answer is 6:36 or can also be written as $\frac{6}{36}$. We can also write this in reduced form. Then our answer becomes 1:6 or $\frac{1}{6}$. You may see probability answers written in reduced form sometimes and at other times not in reduced form.

Example 3:

Sometimes the second event in which we are trying to determine probability depends on what happens in the first event. A candy store has a jar on the counter that contains lollipops. The owner of the store places 10 red lollipops, 5 green lollipops, 5 yellow lollipops and 5 orange lollipops in the jar every morning. Every child can take one free lollipop before leaving the store.

- What is the theoretical probability that the first child will choose a green lollipop?
- Assuming the first child chose a green lollipop, what is the theoretical probability that the second child will choose a red lollipop?
- Assuming the first child chose a green lollipop and the second child chose a red lollipop, what is the theoretical probability that the third child that day will choose a lollipop that is not red?

Solution:

Let's consider the answer to the first question. There are

10 *(red)* + 5 *(green)* + 5 *(yellow)* + 5 *(orange)* = 25 total lollipops.

The first child will choose one of these 25 lollipops of which 5 are green.

The probability of the child picking a green lollipop is $\frac{5}{25}$ or $\frac{1}{5}$. We can also represent this by 5:25 or 1:5.

When the first child takes this one lollipop out of the jar, there are 24 lollipops left. Of these 24 lollipops, 10 are red. The theoretical probability of the second child taking a red lollipop out of the jar is $\frac{10}{24} = \frac{5}{12}$ which can also be represented by 10:24 or 5:12.

When the second child takes this lollipop, there are 23 lollipops left in the jar. Nine of these will be red which means that 14 of them will be other colors. The probability of the third child taking a lollipop that is not red is $\frac{14}{23}$ which can be represented by 14:23.

Now you try one problem.

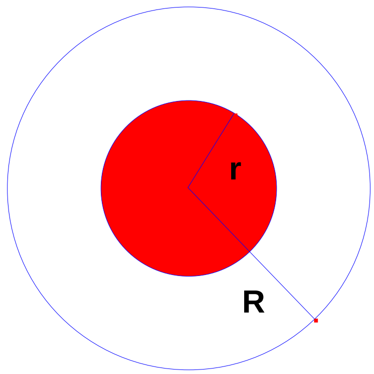
Four children are playing a game that involves a spinner containing three different colors. Half of the spinner is red. One fourth of the spinner is blue and one fourth of the spinner is yellow. What is the probability that one of the children will land on the yellow sector of the spinner? (Hint – draw the spinner to help you.) Show your work in the space provided.

Geometric Probability
Session 2

Geometric probability is the likelihood of an event occurring based on geometric relationships such as area, surface area, or volume.

Example 1:

If an arrow hits the target, what is the probability of hitting the red (shaded) bull's eye?



Solution:

The area of the white circle with radius R is πR^2.

The area of the red (shaded) bulls eye with radius r is πr^2.

Theoretical probability is the likelihood an event will occur under ideal circumstances divided by the total possible outcomes.

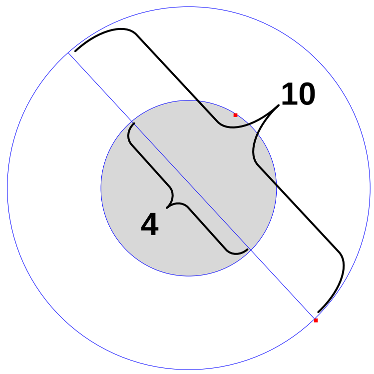
If A represents area, then A_{red} represents the area of the red circle and A_{white} represents the area of the white circle. The probability of hitting the red (shaded) bull's eye is

$$\frac{A_{red}}{A_{white}}$$

If an arrow hits the target, the probability of hitting the red (shaded) bull's eye is $\frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2}$

Example 2:

If an arrow hits the target, what is the probability of hitting the gray (shaded) bull's eye if the diameter of the white circle is 10 and the diameter of the gray circle is 4?



Solution:

The area of the white circle with diameter of 10 is $\pi\left(\frac{10}{2}\right)^2 = 25\pi$.

The area of the gray (shaded) bulls eye with diameter 4 is $\pi\left(\frac{4}{2}\right)^2 = 4\pi$.

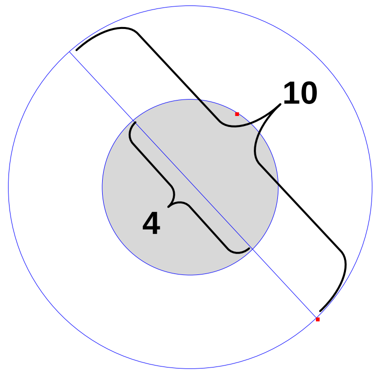
If A represents area, then A_{red} represents the area of the gray circle and A_{white} represents the area of the white circle. The probability of hitting the gray (shaded) bull's eye is

$$\frac{A_{red}}{A_{white}} = \frac{4\pi}{25\pi} = \frac{4}{25}$$

If an arrow hits the target, the probability of hitting the gray (shaded) bull's eye is $\frac{4}{25}$.

Example 3:

If an arrow hits the target, what is the probability of hitting the white part of the target if the diameter of the white circle is 10 and the diameter of the gray circle is 4?



Solution:

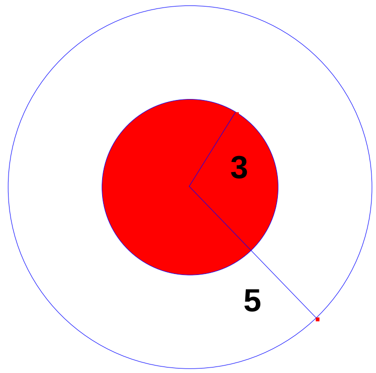
The area of the white circle minus the area of the gray circle is $25\pi - 4\pi = 21\pi$.

The area of the white circle is 25π.

The probability of hitting the white part of the target is $\frac{21\pi}{25\pi} = \frac{21}{25}$.

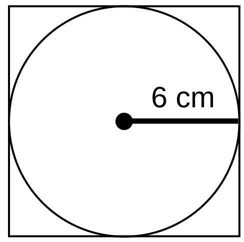
Now you try one:

Problem 1: If an arrow hits the target, what is the probability of hitting the white part of the target if the radius of the white circle is 5 and the radius of the red circle is 3? Show your work in the space provided.



Example 4:

If you choose a point in the square, what is the probability that it is not in the circle?



Solution:

The area of the square is represented by $A = s^2 \Rightarrow A = (2 \bullet 6)^2 = 12^2 = 144\,cm^2$.

The area of the circle is represented by $A = \pi r^2 \Rightarrow A = \pi 6^2 = 36\pi\ cm^2$.

Theoretical probability is the likelihood an event will occur under ideal circumstances divided by the total possible outcomes.

The area of the part of the figure that is not in the circle is $144 - 36\pi$.

The probability that a chosen point is not in the circle is $\frac{144-36\pi}{144} = \frac{36(4-\pi)}{36 \bullet 4} = \frac{4-\pi}{4}$.

What is the probability that a chosen point is in the circle?

Remember that the probability of complementary events must have a sum that totals 1.

- Therefore, the probability that the chosen point is in the circle is:

  $$1 - \frac{4-\pi}{4} = \frac{4}{4} - \frac{4-\pi}{4} = \frac{4-(4-\pi)}{4} = \frac{4-4+\pi}{4} = \frac{\pi}{4}.$$

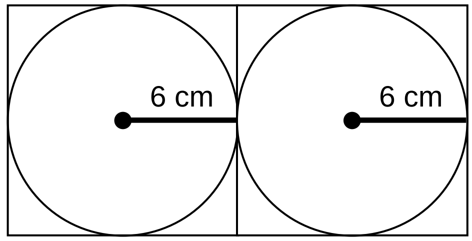
- A second way to calculate the probability is to divide the probability of finding a point in the circle by the probability of finding a point in the square. We can represent this by:

  $$\frac{36\pi}{144} = \frac{\pi}{4}.$$

We obtained the same answer in two different ways.

Example 5:

If you choose a point in the figure, what is the probability that the point is in one of the two circles?



Solution:

The area of the 2 squares is represented by

$A = 2 \bullet s^2 \Rightarrow A = 2 \bullet (2 \bullet 6)^2 = 2 \bullet 12^2 = 2 \bullet 144 = 288$ cm^2.

The area of the 2 circles is represented by $A = 2 \bullet \pi r^2 \Rightarrow A = 2 \bullet \pi 6^2 = 2 \bullet 36\pi = 72\pi$.

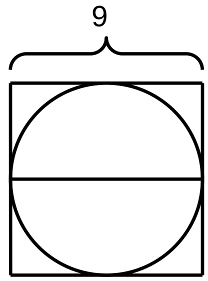
The probability that the point is in the circles is $\frac{72\pi}{288} = \frac{\pi}{4}$.

Note that this is the answer to the same problem with one square and one circle. Do you think this is correct? Justify your answer.

Now you try some.

Problem 2:

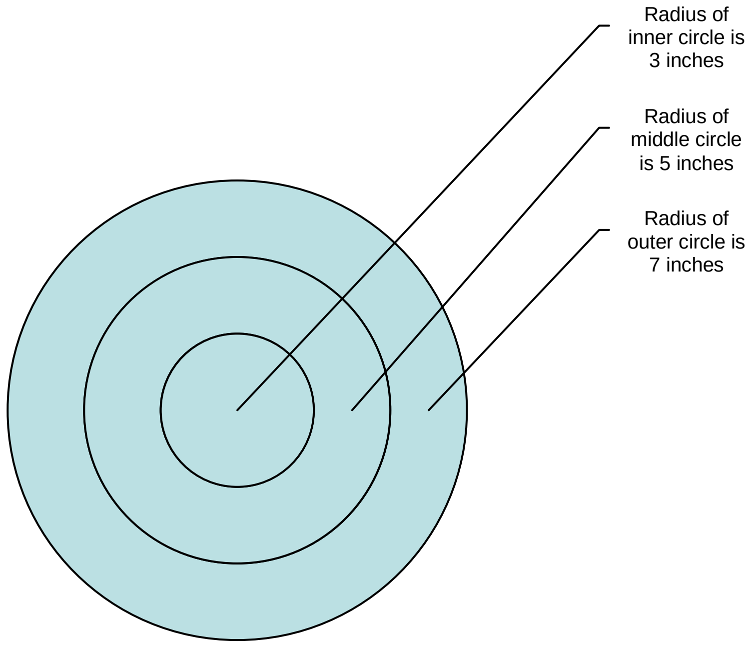
If you choose a point in the square, what is the probability that it is not in the circle? Show your work in the space provided.



Solution:

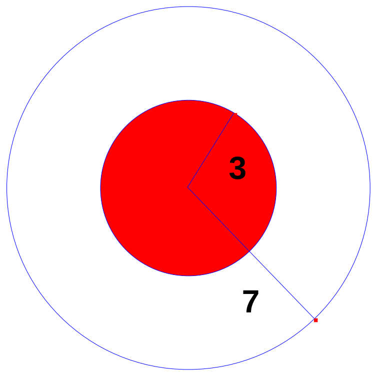
Problem 3:

What is the probability that a dart thrown at the target will land in the middle circle?

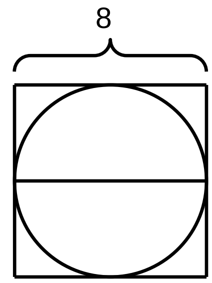


Geometric Probability
Assessment 1

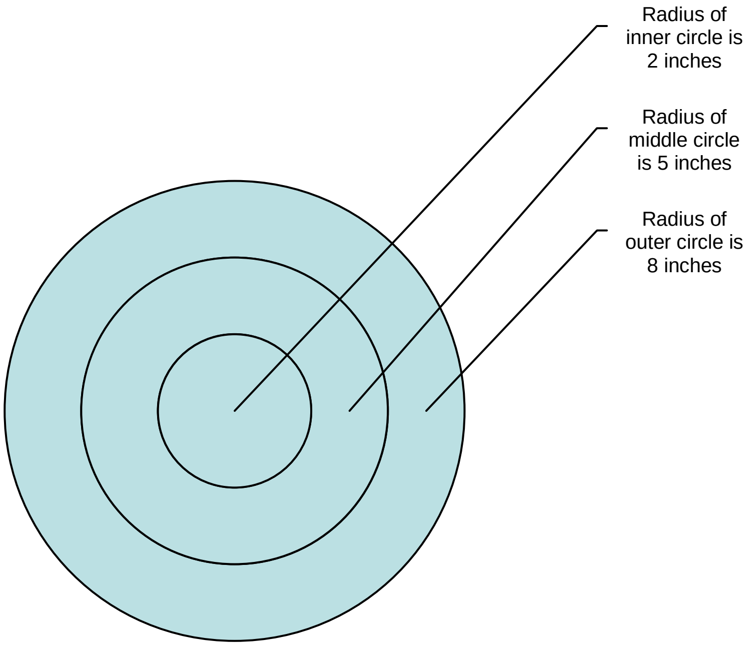
1. If an arrow hits the target, what is the probability of hitting the white part of the target if the radius of the white circle is 7 and the radius of the red circle is 3? Show your work in the space provided.



2. If you choose a point in the square, what is the probability that it is not in the circle? Show your work in the space provided.



3. What is the probability that a dart thrown at the target will land in the outer circle?

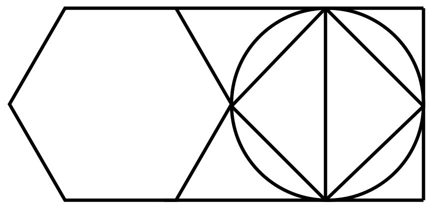


Extensions

1. Determine the probability given a more complicated composite figure. An enrichment activity is included in this lesson.

Geometric Probability
Enrichment

Consider the following composite figure and answer the questions that follow.



1. Describe the shapes that make up the composite figure.

2. Describe how to find the area of every shape that makes up the composite figure that you described in your answer to question 1.

3. Describe the process you would use to find the probability of a choosing a point at random that was not in the circle in the composite figure.

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards, p. 331-333
2008 The Final Report of the National Mathematics Advisory Panel, p. 17
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools, p. 10-17