Arizona Department of Education
8-Week Pandemic Curriculum

Mathematics
Grade 9

This curriculum was designed for self-guided student learning and aligns to Arizona Academic Standards.

Developed October, 2009

Curriculum Overview
Mathematics Grade 9

Lesson Title	Duration	Materials	Assessments (number)
Absolute Value	**Days 1-5** Session 1 (3 hours) Session 2 (2 hours)	• absolute value worksheets	2
Distance and Midpoint	**Days 6-10** Session 1 (1 hour) Session 2 (4 hours)	• midpoint and distance worksheets • ruler • graph paper	4
Fundamentals of Graphing Linear Functions	**Days 11-15** Session 1 (1 hour) Session 2 (2 hours) Session 3 (1 hour) Session 4 (1 hour)	• graphing worksheets • ruler • graph paper	4
Graphing Linear Functions – Slope/Intercept	**Days 16-20** Session 1 (2 hours) Session 2 (2 hours) Session 3 (1 hour)	• graphing worksheets • ruler • graph paper	3
Simplifying Algebraic Expressions	**Days 21-27** Session 1 (4 hours) Session 2 (3 hours)	• algebraic worksheets	2
Solving Systems of Equations	**Days 28-35** Session 1 (1 hour) Session 2 (2 hours) Session 3 (2 hours) Session 4 (2 hours) Session 5 (1 hour)	• systems of equations worksheets • ruler • graph paper	3
Simplifying Radical Expressions	**Days 36-40** Session 1 (1 hour) Session 2 (1 hour) Session 3 (2 hours) Session 4 (1 hour)	• radical expression worksheet • number lines worksheet	3

Absolute Value
An ADE Mathematics Lesson
Days 1-5

Author	ADE Content Specialists
Grade Level	9th grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 1: Number Sense
PO 3. Express that the distance between two numbers is the absolute value of their difference.
Concept 2: Numerical Operations
PO 1. Solve word problems involving absolute value, powers, roots, and scientific notation.
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).

Connects To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.
PO 5. Solve linear equations and equations involving absolute value, with one variable.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.

Overview
The distance between two numbers is the absolute value of their difference.

Purpose
The purpose of this lesson is to solidify understanding of absolute value. This allows you to simplify algebraic expressions and equations that include absolute value.

Materials
- Absolute value worksheets

Objectives
Students will:
- Demonstrate that the distance between two numbers is the absolute value of their difference.
- Solve word problems involving absolute value.

Lesson Components

Prerequisite Skills: This lesson builds on grade 7 and 8 skills of modeling and solving problems involving absolute value. In prior grades, you have also learned to find or estimate the location of numbers on a number line.

Vocabulary: *absolute value, distance between two numbers*

Session 1: Absolute Value and Distance (3 days)
1. Express that the difference between two numbers is the absolute value of their difference.

Session 2: Absolute Value in Context (2 days)
1. Solve word problems involving absolute value.

Assessment

There are two assessments that will help pinpoint misconceptions before moving on to algebraic expressions and equations containing absolute value. Assessments should be completed after each session before moving onto the next session or lesson.

Absolute Value
Session 1 – Absolute Value and Distance

It is often desirable to find the distance between two numbers on a number line. Let's examine some numbers and find the distance between them on the number line.

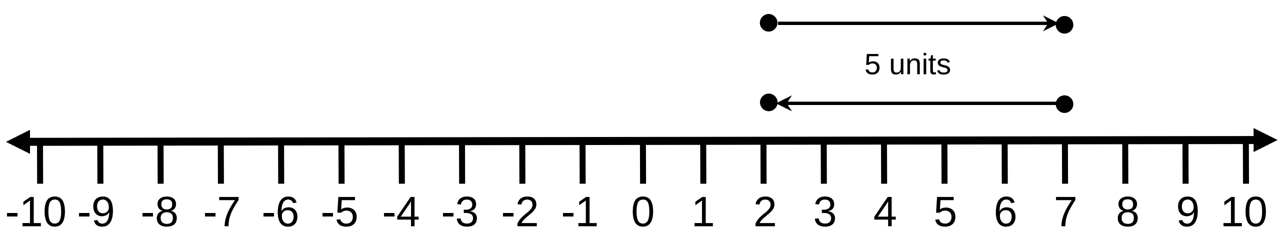
Example 1:

Find the distance between 2 and 7 on the number line.

Solution:

First place the numbers on the number line.

Count the units from 2 to 7 or from 7 to 2. Since we are trying to find distance, and distance is always positive, it does not matter from which endpoint we begin to count.



Since it does not matter in which order we count, 2 to 7 or 7 to 2, we can designate this by using an absolute value sign.

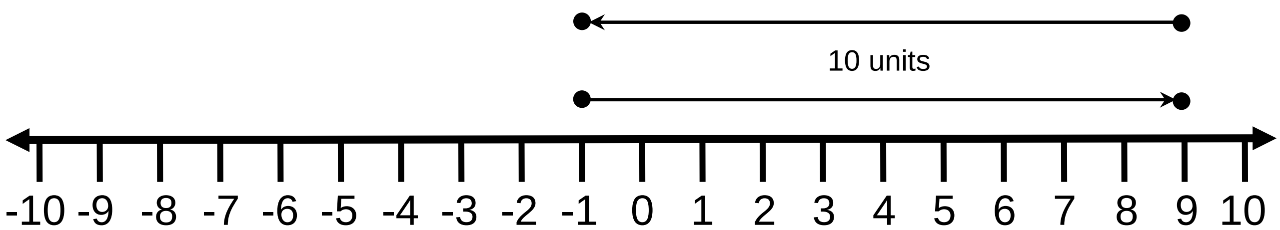
Remember that **absolute value** is defined by a number's distance from zero on a number line (e.g., the absolute value of -4 is 4, the absolute value of 4 is 4; symbolically, $|-4| = 4$ and $|4| = 4$).

- The **distance between two numbers** is the absolute value of their difference.
- The difference between two numbers a and b is $|a-b|$. It makes no difference which number we designate by "a" and which number we designate by "b".

In Example 1, the difference between 2 and 7 is $|2-7| = |-5| = 5$. Note that we can also say that the difference between 2 and 7 is $|7-2| = |5| = 5$. In either case, we find that the difference between 2 and 7 is 5 as shown in our number line above.

Example 2:

Find the distance between 9 and -1 using a number line.



Solution: In Example 2, the difference between 9 and -1 is $|9-(-1)| = |10| = 10$. We could also say that the difference between 9 and -1 is $|-1-9| = |-10| = 10$. In both cases, we find that the difference between 9 and -1 is 10.

Because we are taking the absolute value of the difference of the numbers, we will always end up with a positive number and order will not matter.

It is not always convenient to draw a number line so often we find the distance between two numbers by finding the absolute value of their difference.

Example 3:

Find the distance between 2 and 30.

Solution:

$$|2-30| = |-28| = 28 \textbf{\textit{ or }} |30-2| = |28| = 28$$

It is not necessary to find the distance two ways. This is done to show you that the order does not matter. Can you indicate this problem using a number line in the space provided?

Example 4:

Find the distance between -15 and 29.

Solution:

$$|-15-29| = |-44| = 44$$

Can you find the distance between -15 and 29 in another way in the space provided?

Example 5:

Find the difference between -101 and -123.

Solution:

$$\left|-101-(-123)\right| = \left|-101+123\right| = 22$$

Can you find the distance between -101 and -123 in another way in the space provided?

In questions 1 – 3, draw a number line in the space provided to help you solve the problem.

1. Using a number line, find the distance between 1 and 8. Justify your answer.

2. Using a number line, find the distance between -5 and 12. Justify your answer.

3. Using a number line, find the distance between -10 and -1. Justify your answer.

For the remaining problems, find the difference using absolute value and without using a number line. Show your work in the space provided.

4. Find the distance between 5 and 44.

5. Find the distance between -11 and 78.

6. Find the distance between -53 and 2.

7. Find the distance between 8 and -19.

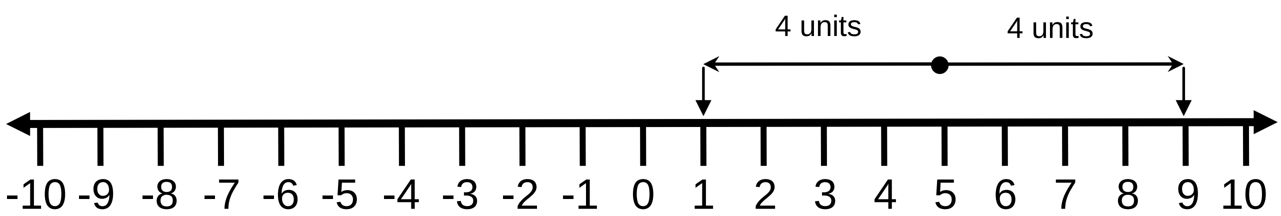
8. Find the distance between -111 and -222.

9. Find the distance between 21 and -1.

10. Find the distance between -30 and -44.

Example 6: The distance between two numbers is 4. One number is 5. What are the possibilities for the other number?

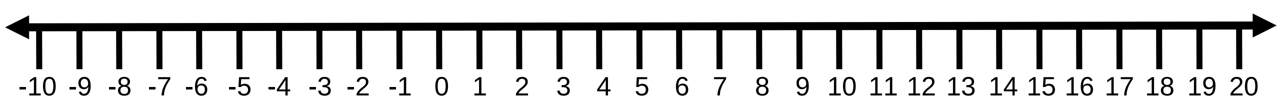
Solution: You may see problems like this. Keep in mind that there are two possible answers for this problem. The distance may be to the right or left of 5. In this case, the other number may be 1 or 9.



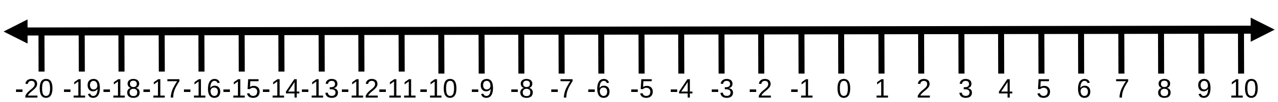
Check: $|9-5|=4$ **or** $|1-5|=|-4|=4$

For the remaining problems, find the other endpoint given an endpoint and length of the line segment. Use a number line to help you.

11. The distance between two numbers is 12. One number is 7. What are the possibilities for the other number?



12. The distance between two numbers is 9. One number is -7. What are the possibilities for the other number?



Absolute Value
Assessment 1

1. Using a number line, find the distance between 1 and 10. Justify your answer.

2. Using a number line, find the distance between -8 and 9. Justify your answer.

3. Without using a number line, find the distance between -5 and -12. Justify your answer.

4. Without using a number line, find the distance between 21 and -19. Justify your answer.

5. Using a number line, find the distance between 120 and -55. Justify your answer.

6. Using a number line, find the distance between -63 and -121. Justify your answer.

Absolute Value
Session 2 – Absolute Value in Context

Many times, word problems involve finding the distance between two quantities. It is necessary to use absolute value to find the distance. Study the following examples.

Example 1:

Determine the difference in altitude between Death Valley (86 meters below sea level) and the summit of Mount St. Helens (2,549 meters above sea level).

Solution:

We can represent the elevation of Death Valley by -86 meters. We can represent the elevation of Mount St. Helens by 2549 meters.

To find the difference between the two elevations, find the absolute value of their difference.

$$|-86-2549| = |-2635| = 2635 \text{ meters}$$

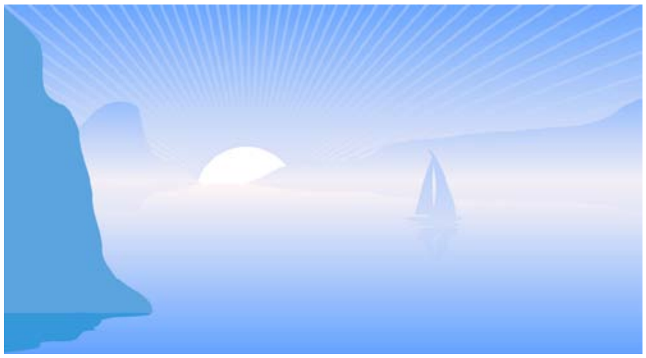
Remember that it does not make any difference in what order the numbers are placed within the absolute value signs.

$$|2549-(-86)| = |2635| = 2635 \text{ meters}$$

We end up with the same answer. The difference between the elevation of Death Valley and Mount St. Helens is 2635 meters.

Example 2:

Determine the difference in altitude between Long Beach, California (7 feet below sea level) and Denver, Colorado (5130 feet above sea level).



Solution:

We can represent the elevation of Long Beach by -7 feet. We can represent the elevation of Denver by 5130 feet.

To find the difference between the two elevations, find the absolute value of their difference.

$$|-7-5130| = |-5137| = 5137 \text{ feet}$$

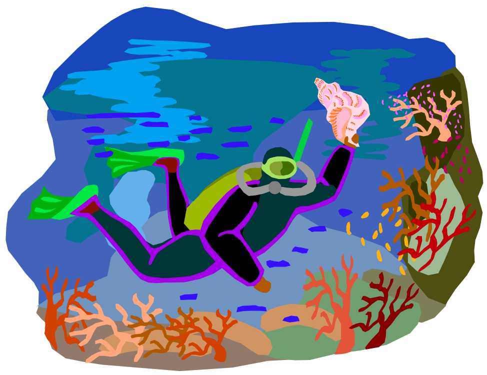
Remember that it does not make any difference in what order the numbers are placed within the absolute value signs.

$$|5130-(-7)| = |5137| = 5137 \text{ feet}$$

We end up with the same answer. The difference between the elevation of Long Beach and Denver is 5137 feet.

Example 3:

On June 13, 2005, a scuba diver, Juno Gomes, reached a depth of 318.25 meters. Many people such as Kazgi Sherpa and Mark Batard have reached the summit of Mt. Everest at 8840 meters. What is the difference in distance between the depth of Gomes' scuba dive and the summit of Mt. Everest?



Solution:

We can represent the dive's depth by -318.25 meters. We can represent the summit of Mount Everest by 8840 meters.

To find the difference between the two elevations, find the absolute value of their difference.

$$|-318.25 - 8840| = |-9158.25| = 9158.25 \text{ meters}$$

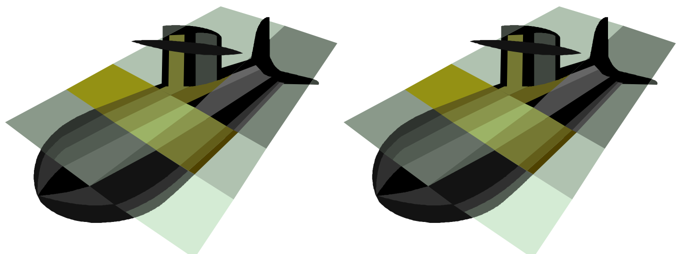
Remember that it does not make any difference in what order the numbers are placed within the absolute value signs.

$$|8840 - (-318.25)| = |9158.25| = 9158.25 \text{ meters}$$

We end up with the same answer. The difference between depth of the dive and the summit of Mt. Everest is 9158.25 meters.

Example 4:

On July 29th, 2008, Russian mini-submarines, Mir-1 and Mir-2, descended 5500 feet to the bottom of Siberia's Lake Baikal. The Concorde airplane, retired on November 26, 2003 rapidly climbed to an altitude of 60,000 feet a very short time in some of its flights. What is the distance between the descent of Mir-1 and Mir-2 and the altitude of Concorde?



Solution:

We can represent the depth of the mini-submarines' dive by -5500 feet. We can represent the altitude of Concorde by 60,000 feet.

To find the difference between the two elevations, find the absolute value of their difference.

$$|-5500 - 60{,}000| = |-65500| = 65{,}500 \text{ feet}$$

Remember that it does not make any difference in what order the numbers are placed within the absolute value signs.

$$|60{,}000 - (-5500)| = |65500| = 65{,}500 \text{ feet}$$

We end up with the same answer. The difference between depth of the dive and the altitude of Concorde is 65,500 feet.

Solve the following problems showing all your work in the space provided.

1. Determine the difference in altitude between Death Valley (86 meters below sea level) and the Alaska's Mount McKinley (6194 meters above sea level).

2. Determine the difference in altitude between New Orleans, Louisiana (8 feet below sea level and Rapid City, South Dakota (3202 feet above sea level).

3. The lowest point in the Dominican Republic is Lago Enriquillo at 46 meters below sea level. The highest point in that country is Pico Duarte at 3175 meters. Determine the difference between the lowest point and the highest point in the Dominican Republic.

4. The lowest point in the Egypt is Qattara Depression at 133 meters below sea level. The highest point in that country is Mount Catherine at 2629 meters. Determine the difference between the lowest point and the highest point in Egypt.

Absolute Value
Assessment 2

Use the information in the following table to answer the questions that follow it.

City or mountain	State or area	Elevation	Feet or Meters	Below or Above Sea Level
Mariana Trench	South Pacific Ocean	10924	meters	below
Death Valley	CA	86	meters	below
Long Beach	CA	7	feet	below
Columbus	OH	685	feet	above
Tucson	AZ	2250	feet	above
Denver	CO	5130	feet	above
Mt. St. Helens	OR	2549	meters	above
Mt. McKinley	AK	6194	meters	above
Mt. Everest	Nepal	8840	meters	above

1. Determine the difference in elevation between Columbus and Tucson.

2. Determine the difference in elevation between Long Beach and Denver.

3. Determine the difference in elevation between the Mariana Trench and Death Valley.

4. Determine the difference in elevation between the Mariana Trench and Mt. Everest.

Extensions

This website provides an activity and applet for you to further explore absolute value on a number line. There is also a link to absolute value functions if you are interested in exploring further.

http://www.analyzemath.com/Definition-Absolute-Value/Definition-Absolute-Value.html

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards
Information for elevations of points:
http://egsc.usgs.gov/isb/pubs/booklets/elvadist/elvadist.html#Highest
http://www.scubarecords.com/
http://en.rian.ru/russia/20080729/115171121.html
http://www.mnteverest.net/history.html
http://classic.mountainzone.com/news/everestspeed.html
http://en.wikipedia.org/wiki/Concorde
http://www.altimeters.net/cityaltitudes2.html
http://www.worldatlas.com/aatlas/infopage/highlow.htm
http://geology.com/records/deepest-part-of-the-ocean.shtml

Distance and Midpoint
An ADE Mathematics Lesson
Days 6-10

Author	ADE Content Specialists
Grade Level	9th grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 1: Number Sense
PO 3. Express that the distance between two numbers is the absolute value of their difference.
Concept 2: Numerical Operations
PO 1. Solve word problems involving absolute value, powers, roots, and scientific notation.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO 1. Determine how to find the midpoint between two points in the coordinate plane.
PO 2. Illustrate the connection between the distance formula and the Pythagorean Theorem.
PO 3. Determine the distance between two points in the coordinate plane.

Strand 5: Structure and Logic
Concept 1: Algorithms and Algorithmic Thinking
PO 1. Select an algorithm that explains a particular mathematical process; determine the purpose of a simple mathematical algorithm.
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO4. Verify characteristics of a given geometric figure using coordinate formulas for distance, midpoint, and slope to confirm parallelism, perpendicularity, and congruence.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Overview
In this lesson, midpoint and distance are explored algebraically and geometrically. You also solve contextual problems involving midpoint and distance. You will study the connection between the distance formula and the Pythagorean Theorem.

Purpose

It is often necessary to solve problems that involve finding either midpoints or distances. It is important to understand exactly what finding the midpoint and distance means. Understanding distance helps you to understand absolute value.

Materials

- Midpoint and distance worksheets
- Ruler
- Graph paper

Objectives

Students will:

- Determine how to find the midpoint between two points.
- Determine how to find the distance on a number line between two values or between two points on a coordinate graph.
- Understand the connection between the Pythagorean Theorem and the Distance Formula.
- Be able to solve contextual problems involving distance or midpoint.

Lesson Components

Prerequisite Skills: This lesson builds upon previous skills of finding rational and irrational points on a number line. Other important skills are applying the meaning of absolute value and graphing points on a coordinate plane.

Vocabulary: *distance, midpoint, Pythagorean Theorem, Midpoint Formula, Distance Formula*

Session 1: Midpoint (1 day)

1. Determine both formally and informally how to find the midpoint between two points. Problems include finding the midpoint given both endpoints, and finding the unknown endpoint given the midpoint and one endpoint.

Session 2: Distance (4 days)

1. Illustrate the connection between the distance formula and the Pythagorean Theorem.
2. Determine the distance between two points in the coordinate plane.

Assessment

There are assessments embedded after each session that pinpoint misconceptions about specific topics.

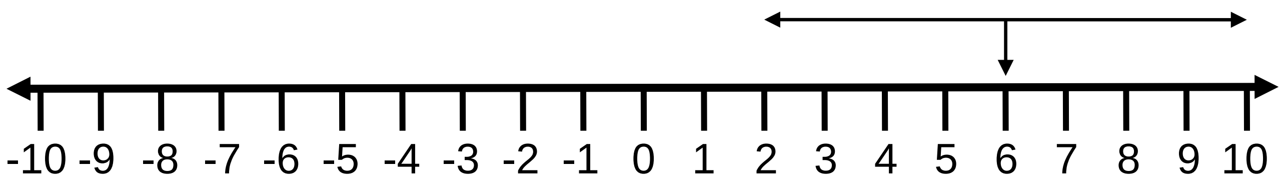
Distance and Midpoint
Session 1 - Midpoint

We can find the midpoint on a number line or the midpoint between two points on a coordinate plane.

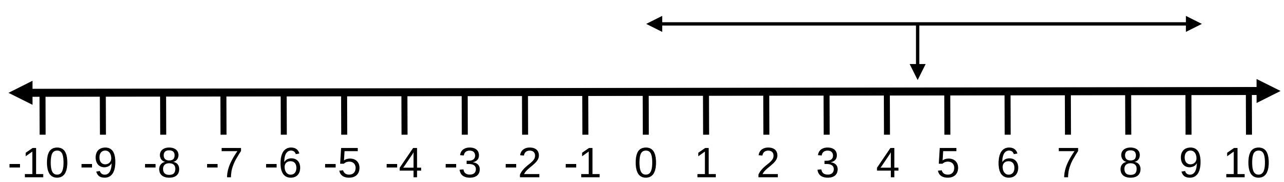
To find the **midpoint** between two values on a number line, find the value that is half-way between the two numbers.

Look at the placements of the numbers on the number line to solve the following problems. The midpoint is the number that is the same distance or equidistant from both numbers on the number line.

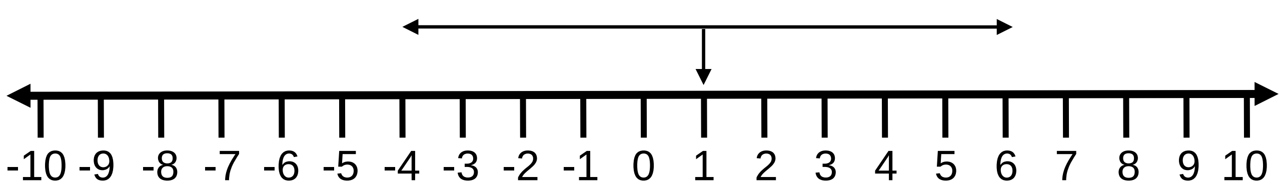
Example 1: Find the midpoint between 2 and 10. If we count from each side the same number of units, we find that the midpoint between 2 and 10 is 6.



Example 2: Find the midpoint between 0 and 9. If we count from each side the same number of units, we find that the midpoint between 0 and 9 is 4.5 or $4\frac{1}{2}$.



Example 3: Find the midpoint between -4 and 6. If we count from the same number of units from each side, we find that the midpoint between -4 and 6 is 1.



We also find the **midpoint** between two numbers by adding the numbers together and dividing the sum by 2.

Let *a* = the first value. Let *b* = the second value. Let *m* = the midpoint.

$$m = \frac{a+b}{2}$$

Do we get the same answer for the examples above if we use this formula?

Example 1: Find the midpoint between 2 and 10.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{2+10}{2} = \frac{12}{2} = 6$$

Example 2: Find the midpoint between 0 and 9.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{0+9}{2} = \frac{9}{2} = 4.5$$

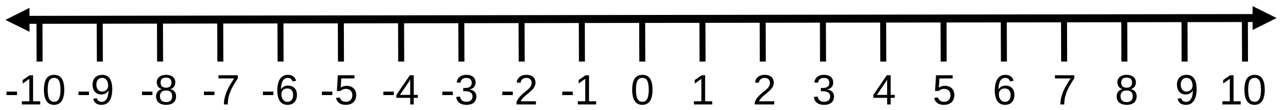
Example 3: Find the midpoint between -4 and 6.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{-4+6}{2} = \frac{2}{2} = 1$$

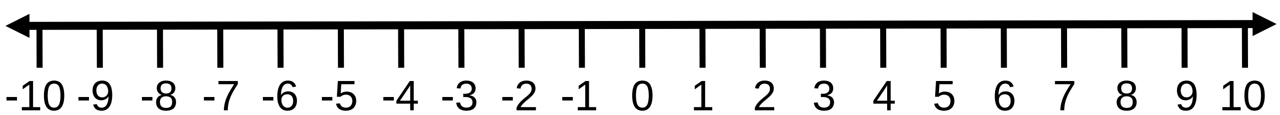
In each example, we arrived at the same midpoint whether we used the number line or the formula to find the midpoint. What is important is that you understand what finding the midpoint between two values means.

Find the midpoint in the following problems by using a number line.

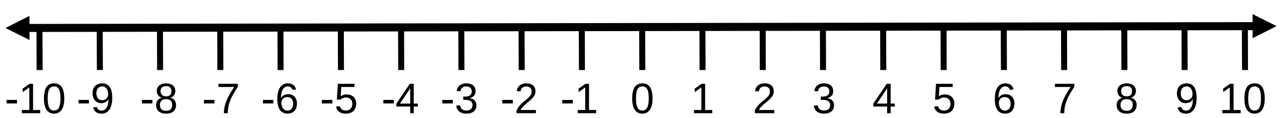
1. Find the midpoint between 1 and 9.



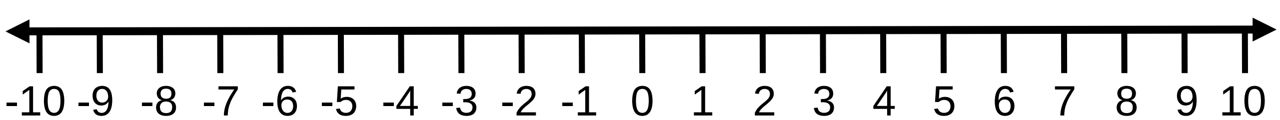
2. Find the midpoint between 0 and 5.



3. Find the midpoint between -6 and 2.



4. Find the midpoint between -8 and -1.



Find the midpoint by using the midpoint formula: $m = \frac{a+b}{2}$. Show all your work.

1. Find the midpoint between 20 and 6.

2. Find the midpoint between 12 and 40.

3. Find the midpoint between -10 and 100.

4. Find the midpoint between -20 and 5.

5. Find the midpoint between -30 and -15.

Sometimes, we are given one endpoint and the midpoint and asked to find the other endpoint. We still use the formula, $m = \frac{a+b}{2}$, and substitute the values in for *m* and *a* or *b*. It does not matter whether we substitute the given endpoint for *a* or *b* as we will obtain the same answer.

Example 1:

The midpoint of two endpoints is 4. If one endpoint is 6, what is the other endpoint?

Solution: Let *m* = 4 and *b* = 6.

$$m = \frac{a+b}{2}$$

$$4 = \frac{a+6}{2}$$

$$4 \bullet 2 = a + 6$$

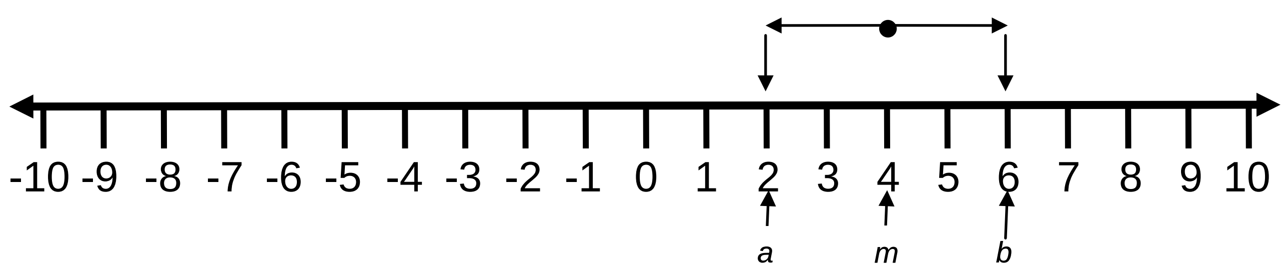
$$8 = a + 6$$

$$a = 2$$

We can check our answer by substituting the value of the endpoints in the midpoint formula to see if we obtain the correct midpoint.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{2+6}{2} = \frac{8}{2} = 4 \text{ (given midpoint)}$$

Note the representation of this problem on the number line.



Note that *b* is 6. This is 2 units to the right of *m,* which is 4. So *a* must be 2 units to the left of 4, which is 2.

Example 2:

The midpoint of two endpoints is -1. If one endpoint is -7, what is the other endpoint?

Solution:

Let *a* = -7 and *m* = -1.

$$m = \frac{a+b}{2}$$

$$-1 = \frac{-7+b}{2}$$

$$-1(2) = -7 + b$$

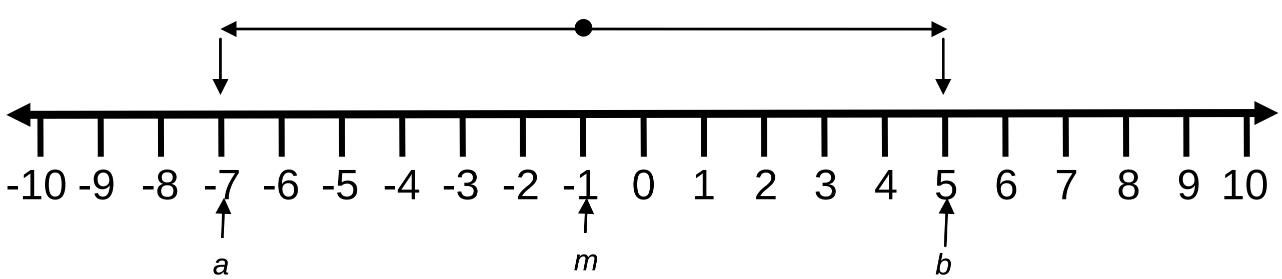
$$-2 = -7 + b$$

$$b = 5$$

We can check our answer by substituting the value of the endpoints in the midpoint formula to see if we obtain the correct midpoint.

$$m = \frac{a+b}{2} \Rightarrow \frac{-7+5}{2} = \frac{-2}{2} = -1 \text{ (given midpoint)}$$

Note the representation of this problem on the number line.



Note that *a* is -7. This is 6 units to the left of *m,* which is -1. So *b* must be 6 units to the right of -1, which is 5.

Solve the following problems showing all your work:

1. The midpoint of two endpoints is 7. If one endpoint is 4, what is the other endpoint?

   Remember that $m = \dfrac{a+b}{2}$.

2. The midpoint of two endpoints is 2. If one endpoint is -5, what is the other endpoint?

Distance and Midpoint
Assessment 1

Find the midpoint of the following pairs of endpoints using the both a number line and the midpoint formula. Show all your work.

1. 2 and 6

2. 4 and 10

3. -3 and -8

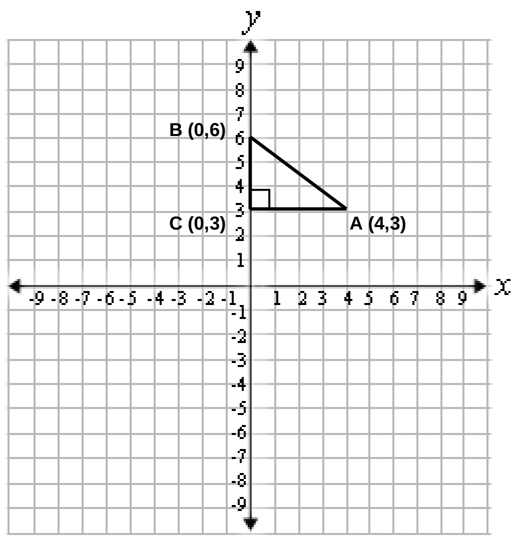
4. -10 and 4

Distance and Midpoint
Session 2 – Distance

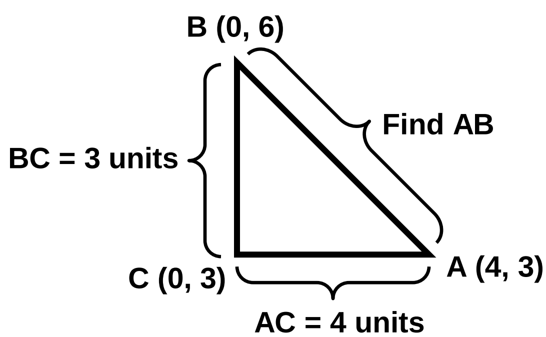
There are different ways to find the distance between two points in the coordinate plane.

Method 1: We can use the Pythagorean Theorem to find the distance between two points.

Example 1: Find the distance between points A and B on the coordinate grid below.



Solution: The vertices of the triangle are A (4, 3), B (0, 6) and C (0, 3). If we count on the coordinate grid, we find the length of AC is 4 units and the length of BC is 3 units. We have a right triangle so we can use the Pythagorean Theorem to find the length of AB.



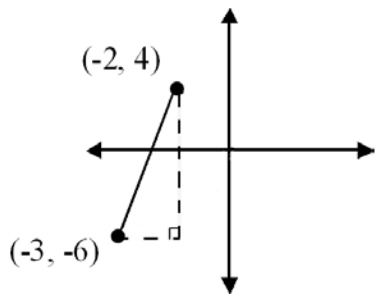
Use the Pythagorean Theorem $c^2 = a^2 + b^2$. Remember that *a* and *b* are the legs of the right triangle and *c* is the hypotenuse. Each side is opposite each angle. Side *a* is 3 units, side *b* is 4 units. Find side *c*.

$$c^2 = a^2 + b^2 \Rightarrow c^2 = 3^2 + 4^2 \Rightarrow c^2 = 9 + 16 = 25 \Rightarrow c = \sqrt{25} = 5.$$

Therefore the length of AB is 5 and the distance between points A and B is 5 units.

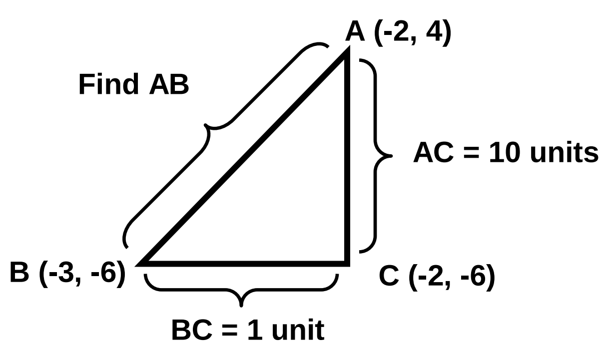
Example 2:

Find the distance between the given points (-2, 4) and (-3, -6) in the diagram given below.



Solution:

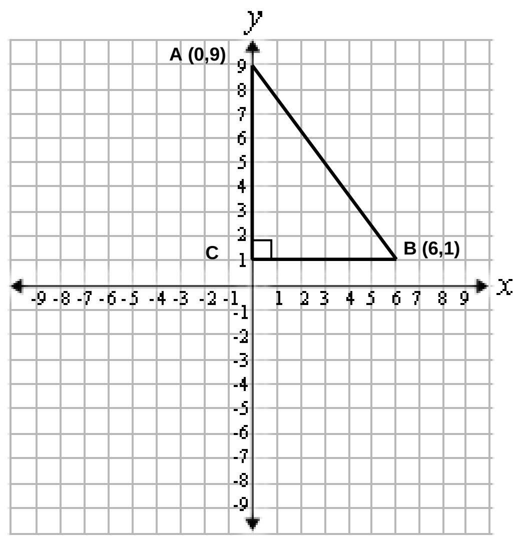
We must first determine the coordinates of the missing point which is the vertex of the right angle. Since this is a right angle, we know that the x-value of the point is the same as (-2, 4) so the x-value is -2. The y-value will be the same as in the point (-3, -6) so the y-value is -6. Therefore the value of the missing point is (-2, -6). Let's redraw this triangle to find the distance between the given points.



Use the Pythagorean Theorem $c^2 = a^2 + b^2 \Rightarrow c^2 = 1^2 + 10^2 = 101 \Rightarrow c = \sqrt{101}$. Therefore the distance between points A and B is $\sqrt{101}$. This is expressed in simplest form. We can estimate this difference by using a calculator to find an approximate value, $\sqrt{101} \approx$ 10.05 units.

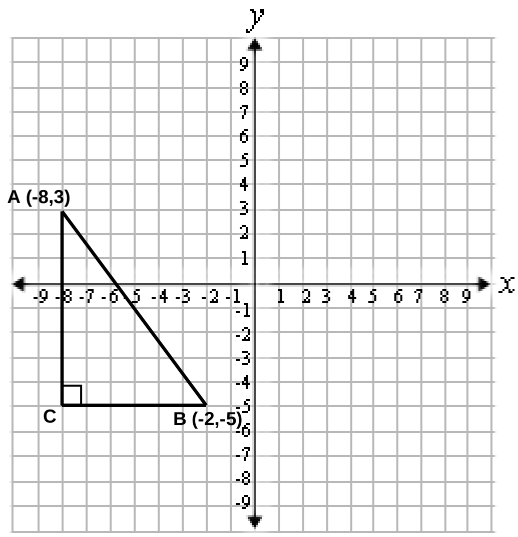
Problem 1:

Find the distance between the given points in the diagram below using the Pythagorean Theorem. Show all your work in the space provided that follows the diagram.



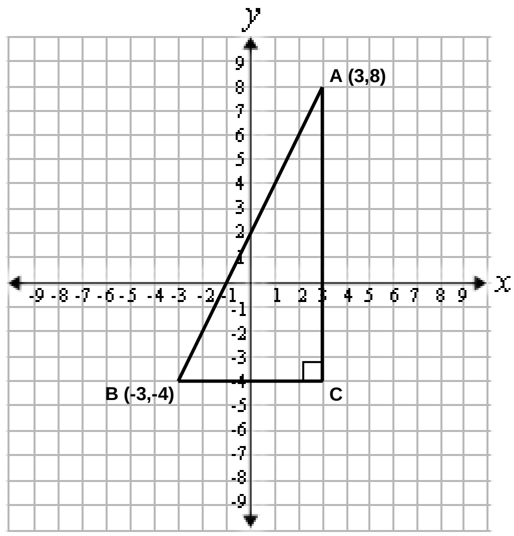
Problem 2:

Find the distance between the given points in the diagram below using the Pythagorean Theorem. Show all your work in the space provided that follows the diagram.

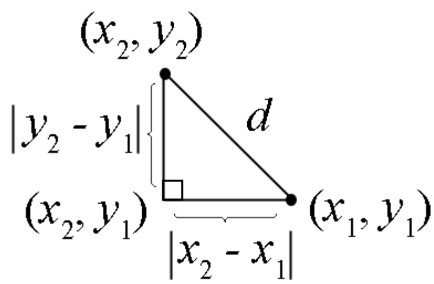


Problem 3:

Find the distance between the given points in the diagram below using the Pythagorean Theorem. Show all your work in the space provided that follows the diagram.



Think about the examples shown previously and the problems that you worked. Then, study the diagram below.



Each coordinate point is labeled. Just as we found the length of each side in each previous triangle, the length of each side is given in this diagram. We are then left to find *d*.

Using the Pythagorean Theorem, we determine that $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ given the vertices of the triangle. In this diagram, we have replaced side "a" by $|x_2 - x_1|$ and side "b" by $|y_2 - y_1|$, and side "c" by *d*.

Then by using the Pythagorean Theorem, we know that $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Taking the square root of each side gives us the <u>distance formula</u>

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **.** Since we are squaring each quantity, the result will be positive and there is no longer a need for the absolute value sign.

The <u>distance formula</u> is a second method for finding the distance between two points on a coordinate grid. Given points (x_1, y_1) and (x_2, y_2), the distance between these points is determined by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Method 2: We can find the distance between the given points, (x_1, y_1) and (x_2, y_2) by using the <u>distance formula</u>.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let's examine one of our previous examples.

Example 1:

Find the distance between the points (4, 3) and (0, 6).

Solution:

(x_1, y_1) = (4, 3) and (x_2, y_2) = (0, 6). Therefore, $x_1 = 4$, $x_2 = 0$, $y_1 = 3$ and $y_2 = 6$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ \Rightarrow \ d = \sqrt{(0-4)^2 + (6-3)^2}$$

$$d = \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$d = 5$$

This is the same answer that we obtained by using the coordinate graph in our previous example 1 under Method 1.

Note – It does not matter which point is designated as (x_1, y_1) and which point is designated by (x_2, y_2), the result will be exactly the same. Try the above example designating the first point as (x_1, y_1) and the second point as (x_2, y_2).

(x_1, y_1) = (0, 6) and (x_2, y_2) = (4, 3). Therefore, $x_1 = 0$, $x_2 = 4$, $y_1 = 6$ and $y_2 = 3$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ \Rightarrow d = \sqrt{(4-0)^2 + (3-6)^2}$$

$$d = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$d = 5$$

This is the same answer that we got when we labeled the two points differently.

Let's clarify our thinking before moving on to more examples using the second method to find the distance between two points on a coordinate plane.

Think about the information, examples, and problems that we have worked and then answer the following questions.

1. Explain the relationship between a right triangle and finding the distance between two points on a coordinate plane.

2. Explain how to find the distance between two points on a coordinate plane by drawing a right triangle and using the Pythagorean Theorem.

3. What is the formula to find the distance between two points on a coordinate grid? Explain briefly how that formula is derived in your own words.

4. Does it matter in which order the points are put into the distance formula? Explain your answer.

Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, find the distance between the given points in the earlier three problems (pages 13-15) you found using the Pythagorean Theorem.

1. Find the distance between the points (0, 9) and (6, 1). Show all your work in the space provided.

2. Find the distance between the points (-8, 3) and (-2, -5). Show all your work in the space provided.

3. Find the distance between the points (3, 8) and (-3, -4). Show all your work in the space provided.

Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, find the distance between the given points in the space provided.

4. Find the distance between the points (2, 4) and (8, -4). Show all your work in the space provided.

5. Find the distance between the points (-3 0) and (-6, -4). Show all your work in the space provided.

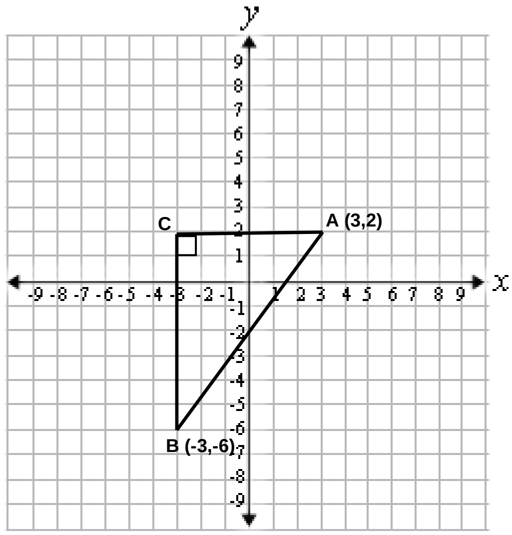
6. Find the distance between the points (10, 7) and (-2, 2). Show all your work in the space provided.

In the space provided below, show that it does not make a difference in what order you place the points in the distance formula. Show all of your work.

- Choose any two points in the coordinate plane.
- Use the distance formula twice.
- The first time assign (x_1, y_1) to the first point you list.
- The second time assign (x_1, y_1) to the second point you list.
- Show that your answers are the same.

Distance and Midpoint
Assessment 2

Find the length of AB on the coordinate plane by using the Pythagorean Theorem. Show all your work in the space provided following the graph.



Distance and Midpoint
Assessment 3

Find the distance between the following set of points using the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Show all your work in the space provided.

1. Find the distance between the points (4, 6) and (-2, -2).

2. Find the distance between the points (9, 5) and (-1, 2).

Distance and Midpoint
Assessment 4

1. Answer the following questions true or false. Justify your answer.

   a. Distance is always positive.

   b. The distance formula that enables us to find the distance between two points on a coordinate plane can be derived from the Pythagorean Theorem.

   c. When trying to use the distance formula, it makes a difference in the correctness of the answer if the first point is not labeled (x_1, y_1) and the second point is not labeled (x_2, y_2).

2. Complete the sentence with the correct phrase(s). Explain your answer.

   a. To find the midpoint of a line segment, _______________________.

   b. To find the distance between two points on a coordinate plane, ____.

3. Explain the relationship between the number line and the midpoint formula to find the midpoint between two sets of points.

Extensions

1. In the extension lesson, you will simplify radicals after applying the distance formula.

Distance and Midpoint
Extension

To find the distance between two points, (x_1, y_1) and (x_2, y_2), use the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,.$$

Distance is always positive.

It is sometimes necessary to simplify the radical after using the distance formula. Study the following examples.

Example 1:

Find the distance between the points (1, -3) and (-5, 0) using the distance formula.

Solution:

Let (x_1, y_1) = (1, -3) and (x_2, y_2) = (-5, 0). Then, $x_1 = 1$, $x_2 = -5$, $y_1 = -3$, and $y_2 = 0$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \;\Rightarrow\; d = \sqrt{(-5-1)^2 + (0-(-3))^2}$$

$$d = \sqrt{(-6)^2 + 3^2} = \sqrt{36+9} = \sqrt{45}$$

$$\sqrt{45} = \sqrt{9 \bullet 5} = 3\sqrt{5}$$

The distance between (1, -3) and (-5, 0) is $3\sqrt{5}$.

Example 2:

Find the distance between the points (6, 4) and (2, -4) using the distance formula.

Solution:

Let (x_1, y_1) = (2, -4) and (x_2, y_2) = (6, 4). Then, $x_1 = 2$, $x_2 = 6$, $y_1 = -4$, and $y_2 = 4$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \;\Rightarrow\; d = \sqrt{(6-2)^2 + (4-(-4))^2}$$

$$d = \sqrt{4^2 + 8^2} = \sqrt{16+64} = \sqrt{80}$$

$$\sqrt{80} = \sqrt{16 \bullet 5} = 4\sqrt{5}$$

The distance between (6, 4) and (2, -4) is $4\sqrt{5}$.

Example 3:

Find the distance between the points (5, 2) and (-2, -5) using the distance formula.

Solution:

Let (x_1, y_1) = (5, 2) and (x_2, y_2) = (-2, -5). Then, $x_1 = 5$, $x_2 = -2$, $y_1 = 2$, and $y_2 = -5$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \;\Rightarrow\; d = \sqrt{(-2-5)^2 + (-5-2)^2}$$

$$d = \sqrt{(-7)^2 + (-7)^2} = \sqrt{49+49} = \sqrt{98}$$

$$\sqrt{98} = \sqrt{49 \bullet 2} = 7\sqrt{2}$$

The distance between the points (5, 2) and (-2, -5) is $7\sqrt{2}$.

Solve the problems below using the distance formula showing all your work in the space provided. Place your answer in simplest radical form.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. Find the distance between the points (4, 0) and (-6, 2).

2. Find the distance between the points (-5, 3) and (5, -2).

3. Find the distance between the points (8, -2) and (4, 10).

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards, p. 290-294 & 308-318
2008 The Final Report of the National Mathematics Advisory Panel, p. 16 & p.28

Fundamentals of Graphing Linear Functions
An ADE Mathematics Lesson
Days 11-15

Author	ADE Content Specialists
Grade Level	9^{th} grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 2: Functions and Relationships
PO 1. Sketch and interpret a graph that models a given context, make connections between the graph and the context, and solve maximum and minimum problems using the graph.
PO 2. Determine if a relationship represented by an equation, graph, table, description, or set of ordered pairs is a function.
PO 4. Use equations, graphs, tables, descriptions, or sets of ordered pairs to express a relationship between two variables.
PO 7. Determine domain and range of a function from an equation, graph, table, description, or set of ordered pairs.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.
PO 3. Write an equation given a table of values, two points on the line, the slope and a point on the line, or the graph of the line.
Concept 4: Analysis of Change
PO 1. Determine the slope and intercepts of the graph of a linear function, interpreting slope as a constant rate of change.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO 5. Graph a linear equation or linear inequality in two variables.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.

Connects To

Mathematics HS:
Strand 2: Data Analysis, Probability, and Discrete Mathematics
Concept 1: Data Analysis (Statistics)
PO 2. Organize collected data into an appropriate graphical representation with or without technology.

Strand 3: Patterns, Algebra, and Functions
Concept 1: Patterns
PO 1. Recognize, describe, and analyze sequences using tables, graphs, words, or symbols; use sequences in modeling.

Strand 4: Geometry and Measurement
Concept 2: Transformation of Shapes
PO 2. Determine the new coordinates of a point when a single transformation is performed on a 2-dimensional figure.
PO 3. Sketch and describe the properties of a 2-dimensional figure that is the result of two or more transformations.
Concept 3: Coordinate Geometry
PO 1. Determine how to find the midpoint between two points in the coordinate plane.
PO 3. Determine the distance between two points in the coordinate plane.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.

Aligns To	Connects To
Mathematics HS: **Strand 5: Structure and Logic** **Concept 2: Logic, Reasoning, Problem Solving, and Proof** **PO 2.** Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s). **PO 3.** Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem. **PO 6.** Synthesize mathematical information from multiple sources to draw a conclusion, make inferences based on mathematical information, evaluate the conclusions of others, analyze a mathematical argument, and recognize flaws or gaps in reasoning.	

Overview

Graphing linear functions is very important. Representing linear functions in several different ways enables you to gain a conceptual understanding of graphing.

Purpose

In this lesson you will graph linear functions by using a table of values or plotting the x- and y-intercept. You will learn how to differentiate functions from relations and how to find the domain and range of a function in different ways.

Materials

- Graphing worksheets
- Ruler
- Graph paper

Objectives

Students will:

- Use equations, graphs, tables, descriptions, or sets of ordered pairs to express a relationship between two variables.
- Graph a linear equation in two variables.
- Determine how to identify functions.
- Determine domain and range of a function from an equation, graph, table, description, or set of ordered pairs.

Lesson Components

Prerequisite Skills: This lesson builds upon prior skills of graphing in a coordinate plane. In grade 6, you graphed ordered pairs in any quadrant of the coordinate plane. In grade 7, you used a table of values to graph an equation or proportional relationship and to describe the graph's characteristics. In grade 8, you determined if a relationship represented by a graph or table is a function.

Vocabulary: *Cartesian coordinate system, coordinate plane, graph, ordered pairs, origin, x-coordinate, y-coordinate, quadrant, axes, element, domain, range, independent variable set, dependent variable set, rule of correspondence, x-intercept, y-intercept, slope, rate of change, relation, function, linear function, vertical line, horizontal line, slope-intercept form of the equation of a line, rise, run*

Session 1 Part 1: Fundamentals of Graphing (1 day)

1. Review graphing points on a coordinate plane.

Session 1 Part 2: Functions (2 days)

1. Determine if a relation is a function.
2. Determine whether a function is a linear function or one of a higher degree.

Session 2 Part 1: Linear Functions and Graphing using a Table of Values or x- and y-intercepts (1 day)

1. Graph linear functions using a table of values or the x-intercept and the y-intercept.

Session 2 Part 2: Domain and Range of a Function (1 day)

1. Determine the domain and range of a linear function.

Assessment

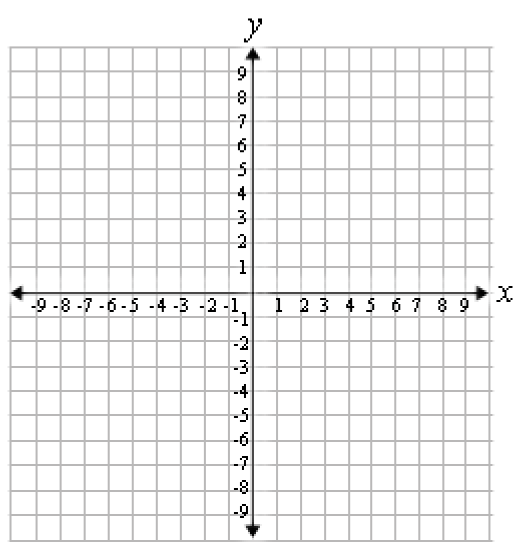
There are multiple assessments scattered throughout the lessons that will help pinpoint misconceptions before moving on to more complex graphs such as the graphs of quadratic functions.

Fundamentals of Graphing Linear Functions
Session 1 Part 1 - Fundamentals of Graphing

Graphing is a technique that is required in many mathematical applications. It is very important to be able to graph straight lines accurately and to understand how to interpret the graph once constructed. To be able to do so requires a good grasp of graphing concepts. This lesson is designed to help you understand graphing concepts from a very elementary level to a more sophisticated level of graphing. It reviews basic graphing techniques and how to determine if a relation is a function.

There are many **fundamental terms** that we need to understand to be able to grasp the concepts. See if you can complete the table with the definition of terms that are associated with graphing. The definitions follow the table so that you can check your work. Note that the definitions should be in your own words.

The Cartesian Coordinate Plane is divided into 4 quadrants. Place the name of each quadrant in its proper location indicating the signs for ordered pairs in that quadrant (+, +) etc. Indicate the origin on the graph.



Does your answer indicate that ordered pairs in Quadrant 1 are (+, +), in Quadrant 2 are (-, +), in Quadrant 3 are (-, -), and in Quadrant 4 are (+, -)? The origin is at (0, 0).

Define the following graphing terms in your own words. Check your answer with the definitions that follow the table.

Term	Definition
ordered pair	
x-intercept	
x-coordinate	
y-intercept	
y-coordinate	
origin	
graph	
quadrant	
axis	
slope of a line	
vertical line	
horizontal line	
Cartesian Coordinate System	

1. **Ordered Pair** - a pair of numbers used to locate and describe points in the coordinate plane in the form (x, y)
2. **x-intercept** - the coordinate at which the graph of a line intersects the x-axis
3. **x-coordinate** - 1st value in an ordered pair
4. **y-intercept** - the coordinate at which the graph of a line intersects the y-axis
5. **y-coordinate** – 2nd value in an ordered pair
6. **Origin** - the intersection of the axes in a coordinate grid, often defined as (0, 0) in two-dimensions
7. **Graph** - a representation of an algebraic equation applied to a coordinate grid
8. **Quadrant** - one of the four sections into which the coordinate plane is divided by the *x*- and *y*-axes
9. **Axis (axes: plural)** (in two-dimensions) - one of two perpendicular number lines used to form a coordinate system
10. **Slope of a line** - the measure of steepness of a line calculated as the change in *y* divided by the change in *x* (the rise over the run)
11. **Vertical Line** – a line perpendicular to the x-axis with an undefined slope ($x = a$)
12. **Horizontal line** – a line parallel to the x-axis with a slope of 0 ($y = b$)
13. **Cartesian Coordinate System** - a plane containing points identified by their distance from the origin in ordered pairs along two perpendicular lines referred to as axes (note: also referred to as coordinate plane and rectangular coordinate plane)

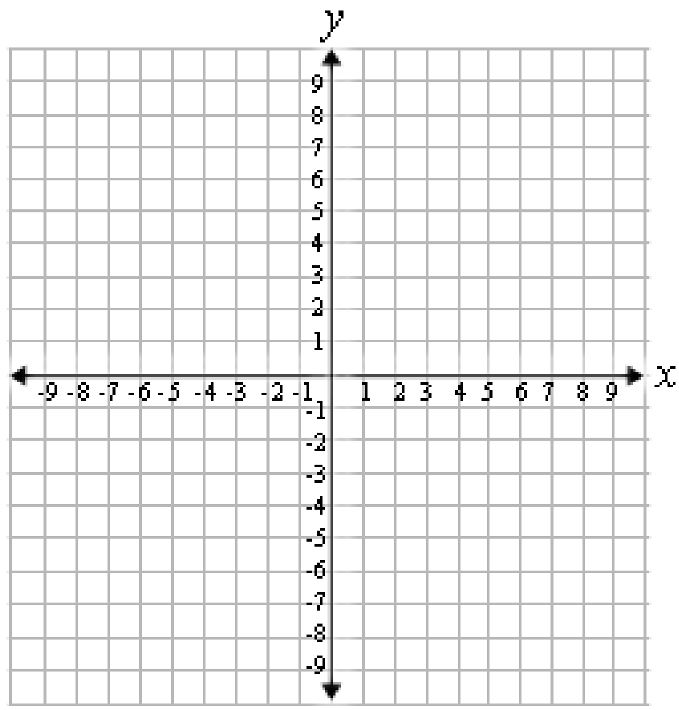
Please note that there are **<u>many terms that describe exactly the same thing</u>** but may vary from textbook to textbook. For example, the following terms are identical in meaning:

- ordered pair, point
- 1st value, x-value, abscissa, input variable
- 2nd value, y-value, ordinate, output variable

To graph a point, the x-value would be indicated on the horizontal axis and the y-value is indicated on the vertical axis. If the x-value is positive, move to the right and if it is negative, move it to the left. If the y-value is positive, move up and if it is negative, move down.

Graph the following points on the graph provided. Place the letter of each point on the graph in the appropriate place.

A (2, 3) B (-1, 8) C (3, -4) D (-7, -7) E (0, 0) F (5, 0) G (-8, -2)



Fundamentals of Graphing Linear Functions
Assessment 1

Define each of the following terms in the space provided.

Term	Definition
relation	
function	
linear function	
quadratic function	
independent variable	
dependent variable	
rule of correspondence	
domain	
range	
slope of a line	
x-intercept	
y-intercept	

Fundamentals of Graphing Linear Functions
Session 1 Part 2 - Functions

A **relation** is a correspondence between a first set, the domain, and a second set, the range, such that each member of the domain corresponds to at least one member of the range. We can express the relationship between two sets in many different ways. Consider the relations:

{4, 2, 0, -2, -4} $\Rightarrow$ {2, 1, 0, -1, -2}

There are 5 **elements** in each set. The first set contains the elements 4, 2, 0, -2, and 4 and is called the **domain** of the relation. The second set contains the elements 2, 1, 0, -1, -2 and is called the **range** of the relation. In this case, each element in the range is one-half of the same element in the domain. This is called a **correspondence**.

Study the following sets and diagrams. Determine why some relations are labeled "yes" and some are labeled "no".

{4, 2, 0, -2, -4} $\Rightarrow$ {2, 1, 0, -1, -2} yes

{1, 2, 3, 4, 5} $\Rightarrow$ {1, 4, 9, 16, 25} yes

{25, 16, 9, 4, 1} $\Rightarrow$ {5, -5, 4, -4, 3, -3, 2, -2, 1, -1} no

{Arizona, Phoenix} $\Rightarrow$ {Diamondbacks, Cardinals, Coyotes, Suns} no

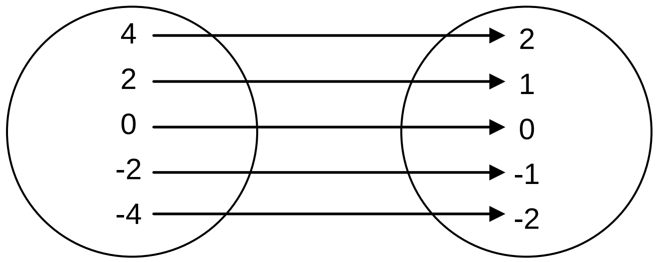
{Arizona, New York} $\Rightarrow$ {Phoenix, Tucson, Buffalo, Albany} no

{1, 1, 2, 2, 3, 3} $\Rightarrow$ {2, 4, 6} yes

Let's look at the correspondence for each relation.

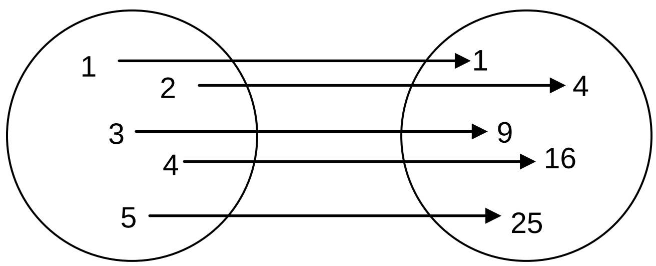
In the first relation, each element in the second set is one-half the corresponding element in the first set.

4⇒2, 2⇒1, 0⇒0, -2⇒-1, -4⇒-2



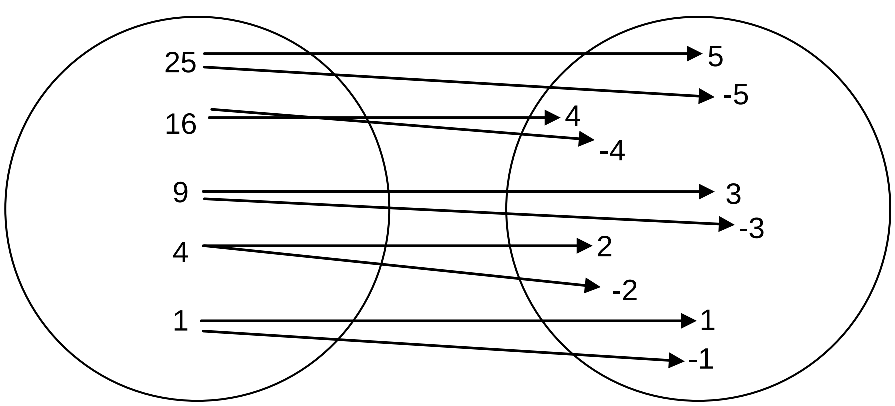
In the second relation, each element in the second set is the square of the corresponding element in the first set.

1⇒1, 2⇒4, 3⇒9, 4⇒16, 5⇒25



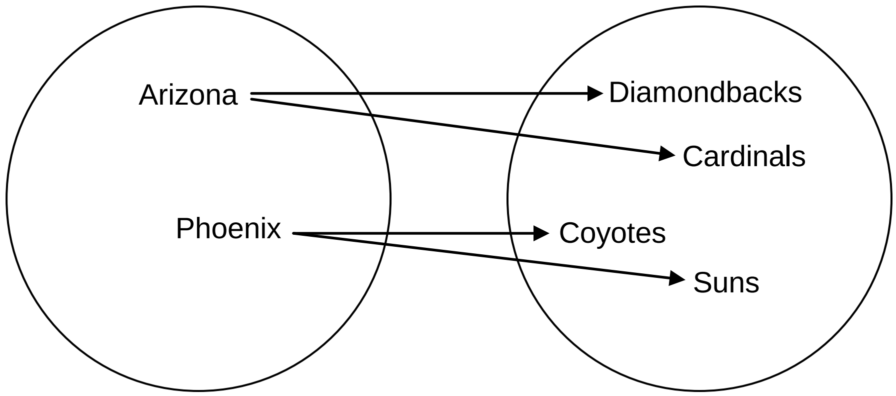
In the third relation, each element in the second set is the square root of the corresponding element in the first set.

25⇒5 and -5, 16⇒4 and-4, 9⇒3 and -3, 4⇒2 and -2, 1⇒-1 and 1



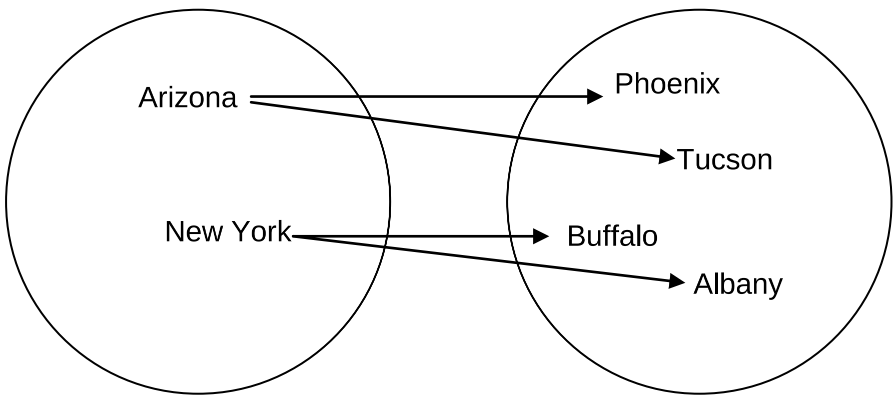
In the fourth relation, each element in the second set is a sports team named by the place in the first set.

Arizona ⇒ Diamondbacks and Cardinals, Phoenix ⇒ Coyotes and Suns



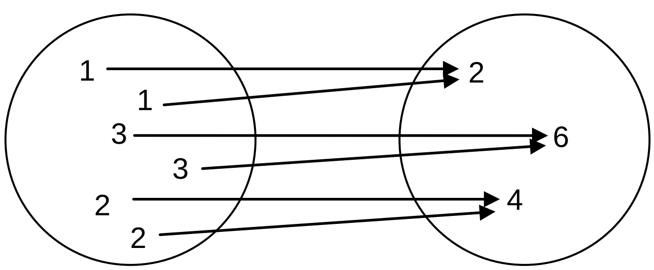
In the fifth relation, each element in the second set is a city located in the state listed in the first set.

Arizona ⇒ Phoenix and Tucson, New York ⇒ Buffalo and Albany



In the sixth relation, each element in the second set is twice that of an element listed in the first set.

1 ⇒ 2, 2 ⇒ 4, 3 ⇒ 6



In each instance the domain is the first set listed and the range is the second set listed. The rule stated in each relation is the correspondence between the first set and the second set. Have you determined why some relations are marked “yes” and some relations are marked “no”? If not, look the example again to help you.

A **function** is a rule that defines a relationship between two sets of numbers in that for each value of the **independent variable set** there is only one value in the **dependent variable set.** Another way to define function is that it is a **correspondence** between a first set, called the **domain**, and a second set, called the **range**, such that each member of the domain corresponds to exactly one member of the range.

The domain is the same as the independent variable set. We can choose any number for the domain. However, because there is some rule or correspondence from the domain to the range, the value of the range is dependent on the values chosen for the domain. Therefore the range is the dependent variable set.

Each of the relations previously listed that had “yes” next to them were functions. Those that had “no” next to them were not functions. For every element in the domain of each relation listed, all those that had only one element in the range were labeled with “yes”. Note that all functions are relations, but not all relations are functions.

Determine which of the following relations are functions and justify your answer. State the correspondence for each relation.

1. {1, 2, 3, 4, 5} $\Rightarrow$ {3, 6, 9, 12, 15}

2. {California} $\Rightarrow$ {Los Angeles, San Francisco, San Diego}

3. {100} $\Rightarrow$ {1, 2, 5, 10, 20, 25, 50, 100}

4. {1, 2, 3, 4, 6, 8, 12, 24} $\Rightarrow$ {24}

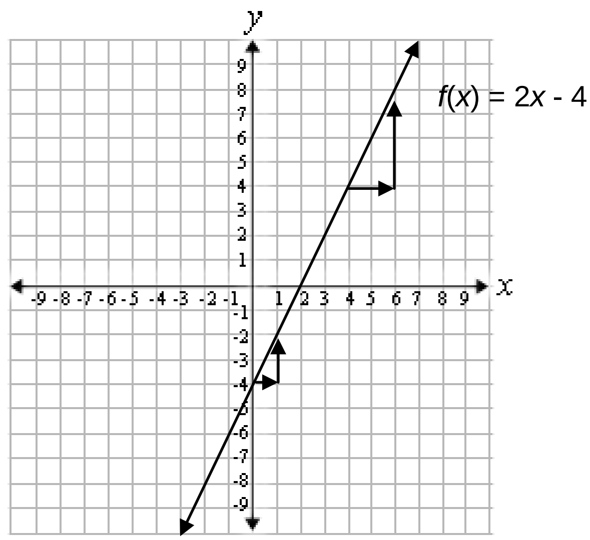
5. {Beagle, Poodle, Chow, German Shepherd} $\Rightarrow$ {Dog}

6. {100, 80, 60, 40, 20} $\Rightarrow$ {5, 4, 3, 2, 1}

Fundamentals of Graphing Linear Functions
Session 2 Part 1 – Linear Functions

Linear functions can be written in the form, $f(x) = mx + b$. A **linear function** is a function that has a constant rate of change and can be modeled by a straight line. In the equation, $f(x) = mx + b$, m is called the slope and b is the y-intercept. An example of a linear function is $f(x) = 2x - 4$.

In this function, the slope is 2 and the y-intercept is -4. Study the graph of $f(x) = 2x - 4$.



The graph of the linear function intercepts the y-axis at (0, -4). The slope is 2 which means for every 1 unit we move horizontally, we move 2 units vertically. Note the slopes indicated on the graph. (4, 4) becomes (6, 8). The change in x is 2 units. The change in y is 4 units. $\frac{4}{2}$ equals 2 which is the slope of the function, $f(x) = 2x - 4$. The point (0, -4) becomes (1, -2). The change in x is 1 unit and the change in y is 2 units. Again, this is the slope of the function, $f(x) = 2x - 4$.

A linear function has no exponent for a variable greater than 1. The function $f(x) = x^2 + 2x + 1$ will be a parabola when graphed as this is a quadratic function.

When the highest exponent of a variable is 2 in a function, it is a quadratic function.

Determine the slope and y-intercept of each linear function.

Function	Slope	y-intercept
$f(x) = 4x + 2$		
$f(x) = -3x - 1$		
$f(x) = -x + 6$		
$f(x) = 7x$		
$f(x) = x - 5$		
$f(x) = 2x + 9$		

Determine if each function is linear or quadratic. Justify your answer.

Function	Linear or Quadratic
$f(x) = 4x - 3$	
$f(x) = x^2$	
$f(x) = x$	
$f(x) = x^2 + 3x - 4$	

Fundamentals of Graphing Linear Functions
Session 2 Part 1
Graphing Using a Table of Values or x- and y-intercepts

When graphing, there are some **expectations** to which we need to adhere. These include:

1. Use graph paper and graph using a pencil.
2. Use a straightedge to make sure that your lines are straight for linear equations.
3. The units on the axes need to be at equal intervals and the units need to be marked so that others reading your graph will know what each tick mark represents on both the x- and y-axes. The units on the x-axis do not need to be the same as the units on the y-axis.
4. The x- and y-axes represent infinite lines so place arrows on the end of each axis.
5. The graph of a linear function is a line not a line segment. Place arrows at the end of the line to indicate the infinite length.
6. Make sure your graph crosses both axes when you extend the line.
7. If you are graphing points, make sure to indicate the points by making them large enough to be visible to the reader. When working with points, it's a good idea to list the coordinates of the point near the graph of the point.
8. Mark the equation of the line on or near the line.
9. If you graph the equation correctly, it should look exactly the same as another person graphing the same equation.
10. Do not graph more than one equation on a set of axes unless the problem asks you to do so. If you are graphing more than one line on the same set of axes, color code or use a legend to indicate which line is the graph of which equation.

There are many **<u>different approaches to graphing an equation</u>**. Let's review some of them.

1. **We can graph an equation by using a table of values.** Two points determine a line but it is good to have three points and use one as a check. You may choose any x value (or y value) and solve for the other value. Then make up a table and graph these points. Now, draw a line connecting these three points indicating the infinite length with arrows.

2. **We can graph an equation by using the x- and y-intercepts.** When using this method, we find the x-value when y is 0 (this is the x-intercept). Then, find the y-value when x is 0 (this is the y-intercept). Graph these two points and draw a line connecting them. Extend the line and draw arrows on each end to indicate the infinite length.

3. **We can graph an equation by placing it in slope-intercept form:** $y = mx + b$ **or** $f(x) = mx + b$. The *m* represents the slope and the *b* is the y-intercept. We first draw a point where the line is going to intercept the y-axis. From this point, we use $\frac{rise}{run}$ to determine a second point. If the run is positive, we move to the right from the y-intercept. If it is negative, we move to the left. If the rise is positive, we move up and if it's negative, we move down. You may do the rise first or the run first and then do the other from where the new point would be.

Example 1: Graph $f(x) = x + 3$

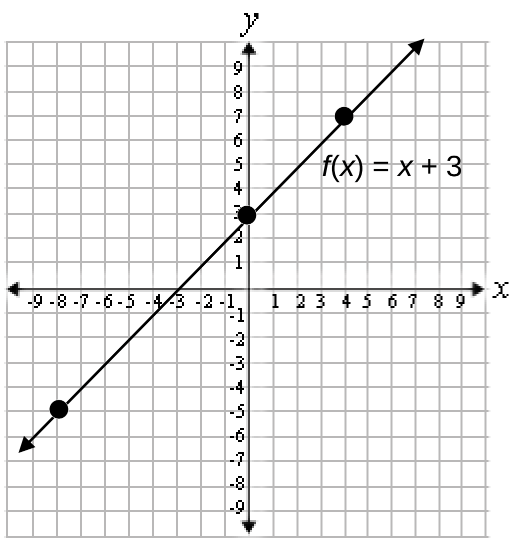
Make sure to follow all the expectations enumerated above.

1. Graph $f(x) = x + 3$ using a table of values:

   a. Find at least three points that satisfy the equation. Two points determine a line but the third point is a check for accuracy. Make a table. You may choose any points you like but remember that you need an appropriate scale for the axes.

x	$f(x)$
0	3
4	7
-8	-5

   b. Graph these points and connect them using the representation of an infinite line.



   c. Label the line appropriately.

2. Graph $f(x) = x + 3$ using the intercepts:

   a. Substitute 0 in for *x* in the equation and solve for $f(x)$. This is the y-intercept (the place where the function crosses the y-axis).

$$f(x) = x + 3 \;\Rightarrow\; f(x) = 0 + 3 \;\Rightarrow\; f(x) = 3$$

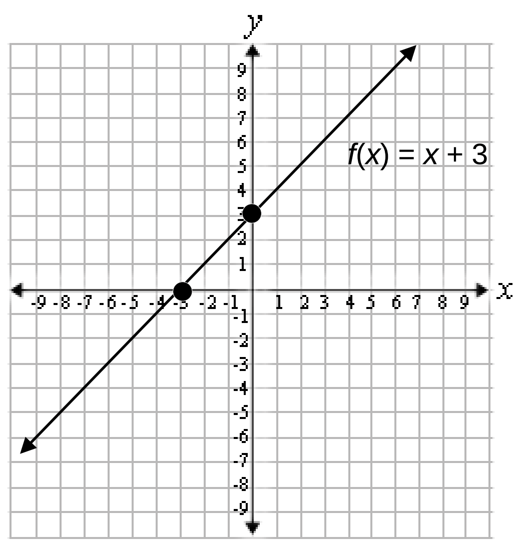
   b. Substitute 0 in for $f(x)$ in the equation and solve for *x*. This is the x-intercept (the place where the function crosses the x-axis).

$$f(x) = x + 3 \;\Rightarrow\; 0 = x + 3 \Rightarrow\; x = -3$$

   c. Plot these two points.

x	$f(x)$	intercept
0	3	y-intercept
-3	0	x-intercept

   d. Draw a line connecting the intercepts using the representation of an infinite line.



   e. Label the line appropriately.

Example 2: Graph $f(x) = 2x - 4$

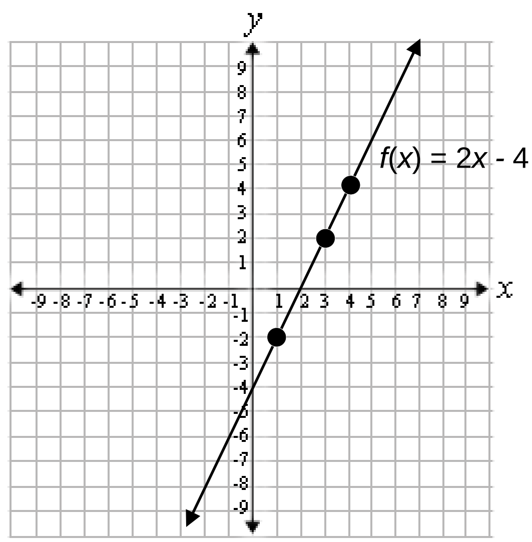
Make sure to follow all the expectations enumerated above.

1. Graph $f(x) = 2x - 4$ using a table of values:

    a. Find at least three points that satisfy the equation. Two points determine a line but the third point is a check for accuracy. Make a table. You may choose any points you like but remember that you need an appropriate scale for the axes.

x	$f(x)$
1	-2
4	4
3	2

    b. Graph these points and connect them using the representation of an infinite line.



    c. Label the line appropriately.

2. Graph $f(x) = 2x - 4$ using the intercepts:

   a. Substitute 0 in for *x* in the equation and solve for $f(x)$. This is the y-intercept (the place where the function crosses the y-axis).

$$f(x) = 2x - 4 \Rightarrow \ f(x) = 2(0) - 4 \ \Rightarrow \ f(x) = -4$$

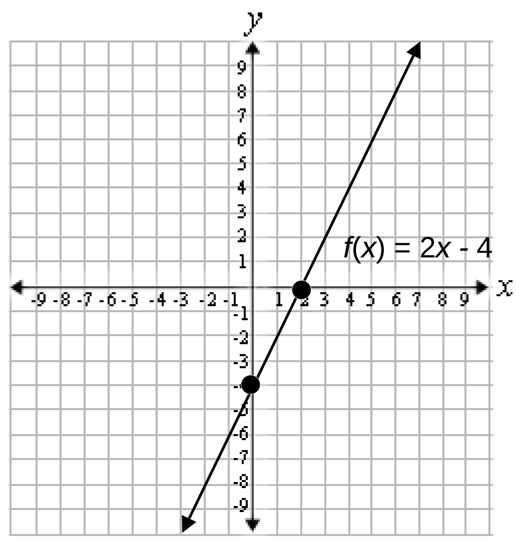
   b. Substitute 0 in for $f(x)$ in the equation and solve for *x* . This is the x-intercept (the place where the function crosses the x-axis).

$$f(x) = 2x - 4 \Rightarrow \ 0 = 2x - 4 \Rightarrow \ 2x = 4 \Rightarrow x = 2$$

   c. Plot these two points.

x	$f(x)$	intercept
0	-4	y-intercept
2	0	x-intercept

   d. Draw a line connecting the intercepts using the representation of an infinite line.

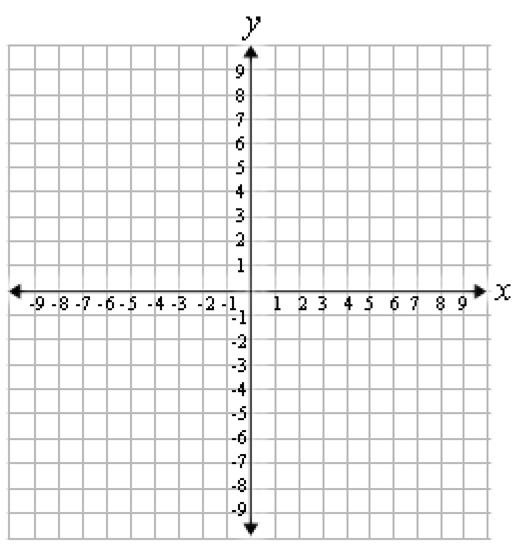


   e. Label the line appropriately.

Now try to graph other functions using both a table of values and the x- and y-intercepts. Remember that in each case, the graph should be identical. All points that are on the first graph made with the table of values should also be on the graph that used the x and y intercepts. Show all your work.

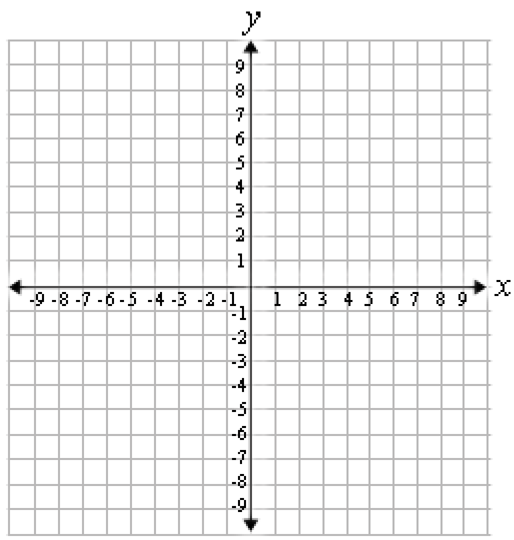
Problem 1a: Graph $f(x) = -x + 2$. Use a table of values.

x	$f(x)$



Problem 1b: Graph $f(x) = -x + 2$. Use the x-intercept and y-intercept.

x	$f(x)$	intercept
		y-intercept
		x-intercept



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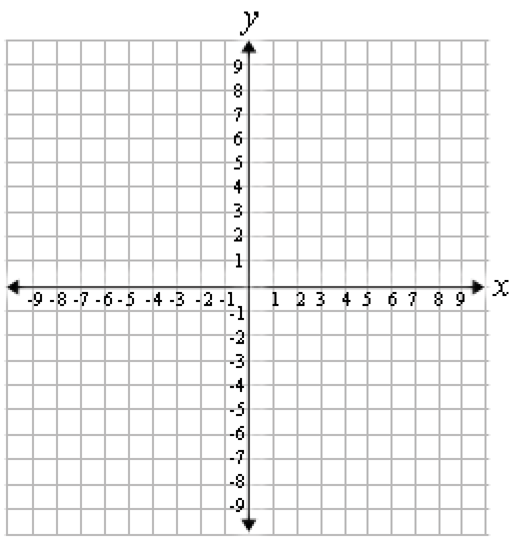
Problem 2a: Graph $f(x) = 3x + 6$. Use a table of values.

x	$f(x)$
-1	
-2	
-3	



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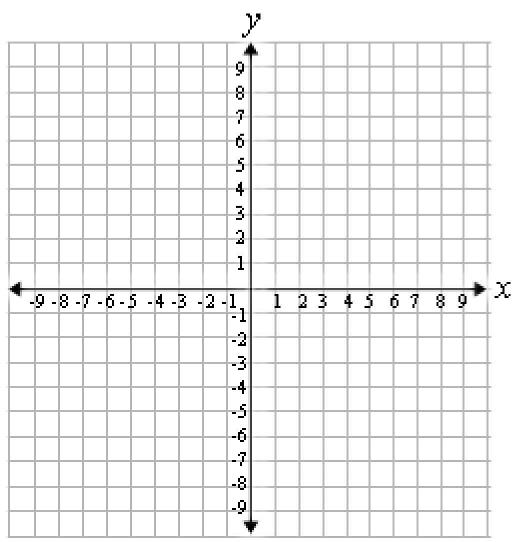
Problem 2b: Graph $f(x) = 3x + 6$. Use the x-intercept and y-intercept.

x	$f(x)$	intercept
		y-intercept
		x-intercept



Fundamentals of Graphing Linear Functions
Assessment 2

Answer the following short answer questions about the fundamentals of graphing linear functions. State whether the following statements are true or false. If false, state why.

1. All relations are functions.

2. The domain of a function is the set of dependent variables.

3. The range of a function is the set of output variables for a function.

4. The x-intercept is the point (0, y).

5. A function is a relation that has exactly one y-value for every x-value.

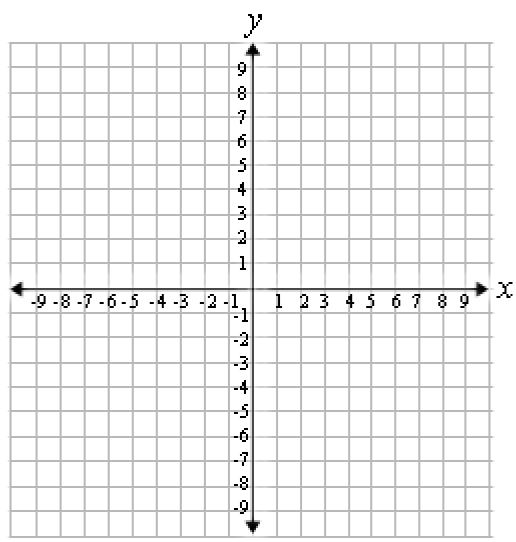
Complete the following sentences with the correct word(s).

6. The ______________variable is the same as the x-value of an ordered pair.
7. The _____________ of the absolute value function, $f(x) = |x|$ is the set of all real numbers that is equal to or greater than 0.
8. Graphs in the Cartesian Coordinate System are divided into four areas. Each area is called a ________________.
9. To graph a function using a table of values, ____.
10. To graph a function using the x-intercept and the y-intercept, ____.

Fundamentals of Graphing Linear Functions
Assessment 3

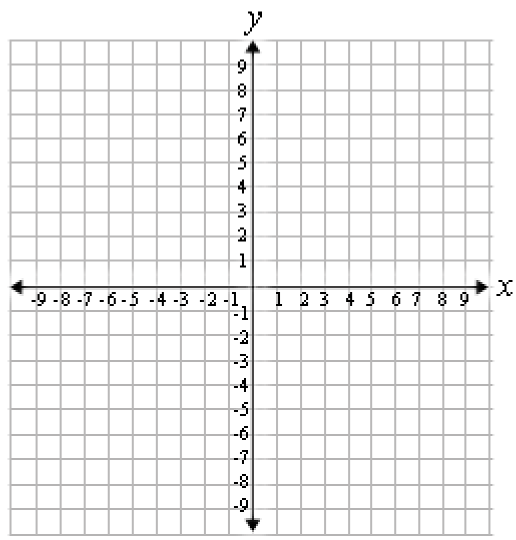
1. Graph $f(x) = 3x - 3$ using a table of values.

x	$f(x)$



2. Graph $f(x) = -2x + 8$ using the x and y-intercepts. Show your work.

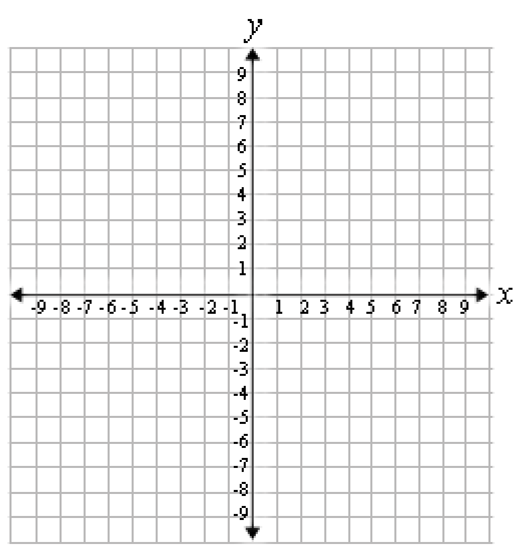
x	$f(x)$	intercept
		y-intercept
		x-intercept



3. Graph $f(x)=\frac{1}{2}x+5$ using a table of values and using the x and y-intercepts.

x	$f(x)$

x	$f(x)$	intercept
		y-intercept
		x-intercept



Fundamentals of Graphing Linear Functions
Session 2 Part 2 – Domain and Range of a Function

The **domain of a function** is the set of values for the independent variable (input value) of a function.

The **range of a function** is the set of all possible output values for a function.

For the purposes of this lesson we will only talk about linear functions.

Let's study some examples of domain and range of a function containing ordered pairs. Each example given below is a function as every x-value goes to exactly one y-value.

Example 1:

Find the domain and range of the function: {(1, 3), (2, 6), (3, 9), (4, 12), (5, 15)}.

Solution:

The set of values for the independent variable is the set of all first values of the ordered pairs. When we choose a first value (input value), we have many choices. Since functions have a rule of correspondence, the second value (output value), is dependent on our first value. Therefore, we call the first or input values the independent variable of the function and the set of second or output values the dependent variable of the function. The set of all input values is the **domain of a function**. The set of all output values is the **range of a function**.

The set of x-values or input values of the set of ordered pairs is {1, 2, 3, 4, 5}. This is the **domain of our function**.

The set of y-values or output values of the set of ordered pairs is {3, 6, 9, 12, 15}. This is the **range of our function**.

Example 2:

Find the domain and range of the function: {(1, 1), (2, 1), (3, 1), (4, 2), (5, 2), (6, 2)}.

Solution:

The set of x-values or input values of the set of ordered pairs is {1, 2, 3, 4, 5, 6}. This is the **domain of our function**.

The set of y-values or output values of the set of ordered pairs is {1, 2}. This is the **range of our function**. We do not write 1 and 2 more than one time even though they appear as y-values more than once.

Example 3:

Find the domain, range, and rule of correspondence for the function listed below.

$$\{\left\{\left(\frac{1}{2},2\right),\left(\frac{1}{3},3\right),\left(\frac{1}{4},4\right),\left(\frac{1}{5},5\right)\right\}$$

Solution:

The set of x-values or input values of the set of ordered pairs is $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$. This is the **domain of the function**.

The set of y-values or output values of the set of ordered pairs is {2, 3, 4, 5}. This is the **range of the function**.

What is the relationship between the x-value and the y-value in this function? They are reciprocals of each other. Remember that this is called the ***rule of correspondence***.

To find the domain and range of a function:

- **List all the x-values (1^{st} values) or independent variable. This is the <u>domain</u> of the function.**
- **List all the y-values (2^{nd} values) or dependent variable. This is the <u>range</u> of the function.**

Each set of ordered pairs below is a function. Study them and in the work space provided find the domain and range of the function, describe the rule of correspondence, and justify your answers.

1. {(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)}

2. {(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36)}

3. {(4, 9), (8, 17), (12, 25), (16, 33)}

4. {(3, 3), (-3, 3), (8, 8), (-8, 8), (11, 11), (-11, 11), (20, 20), (-20, 20)}

The domain and range of a function can also be discovered through a table of values, its equation, or its graph.

For a function represented by a table, follow the same procedure that we used to determine the domain and range of a set of ordered pairs.

To find the domain and range of a function:

- **List all the x-values (1^{st} values) or independent variable. This is the <u>domain</u> of the function.**
- **List all the y-values (2^{nd} values) or dependent variable. This is the <u>range</u> of the function.**

Example 1:

Determine the domain and range of the following function. Explain your reasoning.

x	y
2	-2
4	-4
-6	6
-8	8

Solution:

The set of all x-values is the **domain**: {2, 4, -6, -8}.

The set of all y-values {-2, -4, 6, 8} is the **range**.

Example 2:

What is the domain and the range of $f(x) = x + 3$? Explain your answer.

Solution:

The domain of $f(x) = x + 3$ is the set of all real numbers as we can choose any real number for x.

The range is also the set of all real numbers as all real numbers will result when we choose any number for x.

Example 3:

What is the domain and range of $f(x) = |x|$? Explain your answer.

Solution:

The domain of $f(x) = |x|$ is all real numbers as we can choose any number for x. However, $f(x)$ will always be either 0 or a positive real number so the range of $f(x) = |x|$ will be 0 and the set of positive real numbers. We can represent this by $f(x) \geq 0$.

Example 4:

Find the domain and range of $f(x) = 4$. Justify your answer.

Solution:

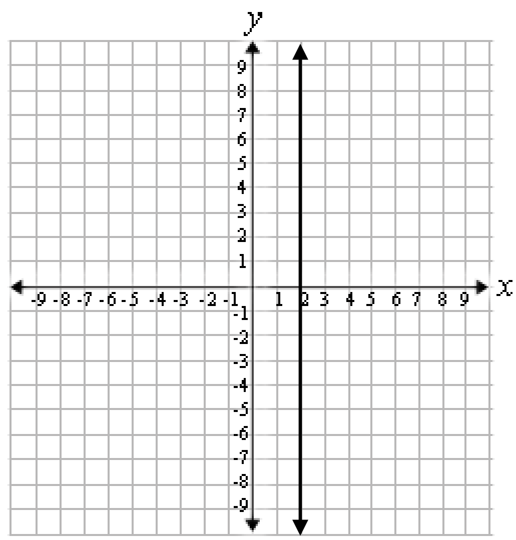
The domain of $f(x) = 4$ is all real numbers as we can choose any number for *x*.

The range of $f(x) = 4$ will always be 4 as for any value we choose for *x*, $f(x)$ will always equal 4.

Example 5:

The graph of $x = 2$ is shown below. Determine the domain and range of $x = 2$ using the graph. Justify your answer.

Please note that $x = 2$ is not a function. For every *x*-value shown (which is 2), *f*(*x*) can be anything. So there is not only one y-value for every x-value.



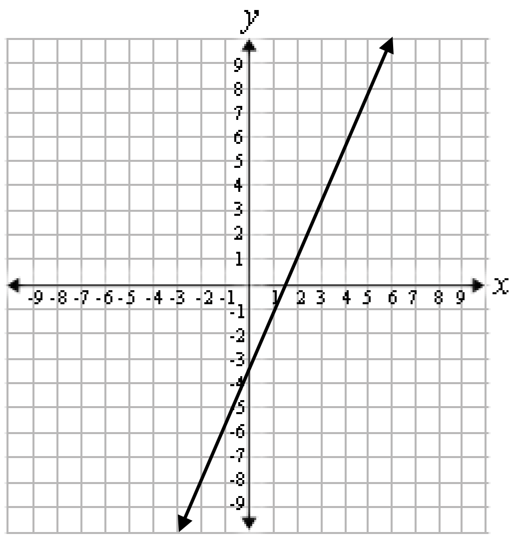
Solution:

The domain of $x = 2$ is {2}.

The range of $x = 2$ is all real numbers as for every x value, y can be any number. For example, {(2, 1), (2, 7), (2, 0), (2, -9)…)} all are ordered pairs on the graph.

Example 6:

Find the domain and range of the function $f(x) = 2x - 4$ represented on the graph below. Justify your answer.



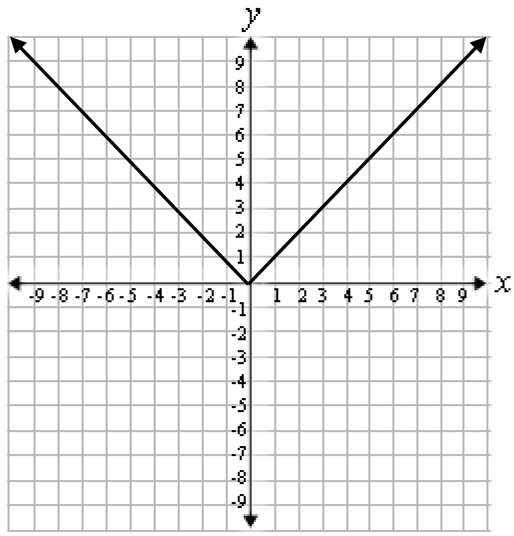
Solution:

The domain for the function $f(x) = 2x - 4$ is all real numbers as any real number may be substituted for x.

The range for the function is all real numbers as once we substitute in a value for x, f(x) will be all real numbers.

Example 7:

Find the domain and range of the function $f(x) = |x|$ represented on the graph below.



Solution:

The domain of the function $f(x) = |x|$ is all real numbers as all real numbers can be used for x-values in the function.

Looking at the graph, we note that the set of ordered pairs satisfying the function will always be in the 1st or 2nd quadrant where the y-value is always positive. Therefore, the range of the function $f(x) = |x|$ is all real numbers greater than or equal to 0 which can be represented by $f(x) \geq 0$.

Determine the domain and range of each of the following in the work space provided. Explain your reasoning.

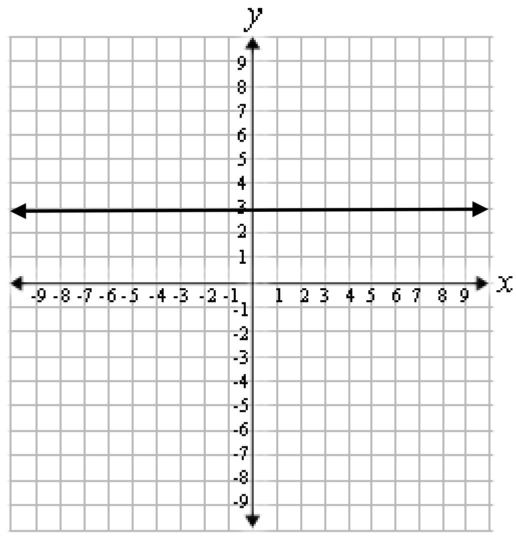
1.

x	y
2	2
12	3
22	4
32	5

2. $f(x) = -x + 5$

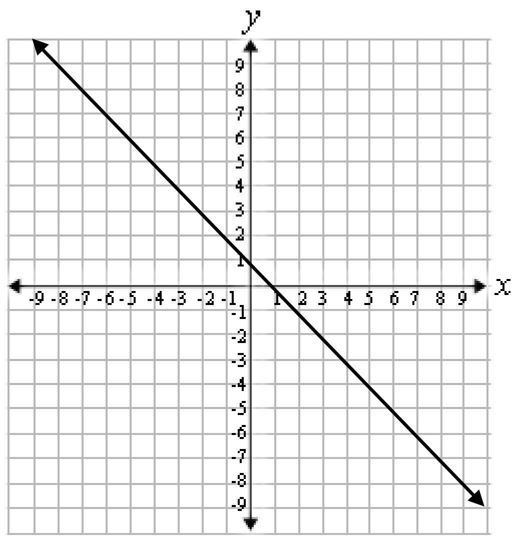
3. $f(x) = 2|x|$

4. $f(x) = 3$



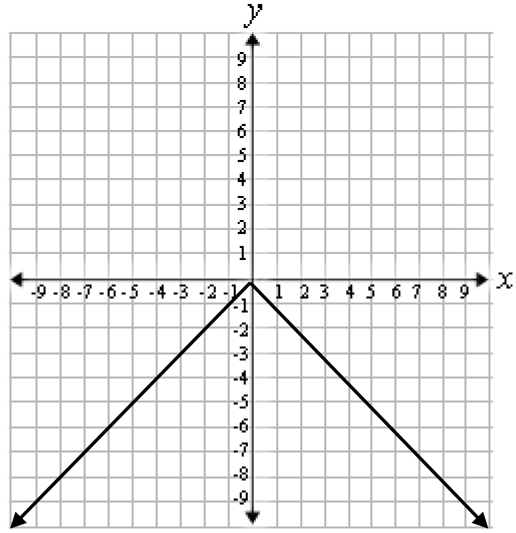
y
x
9
8
7
6
5
4
3
2
1
-9 -8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8 9
-1
-2
-3
-4
-5
-6
-7
-8
-9

5. $f(x) = -x + 1$



y
x
9
8
7
6
5
4
3
2
1
-9 -8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8 9
-1
-2
-3
-4
-5
-6
-7
-8
-9

6. $f(x) = -|x|$



Fundamentals of Graphing Linear Functions
Assessment 4

Determine the domain and range of each of the following in the work space provided. Explain your reasoning.

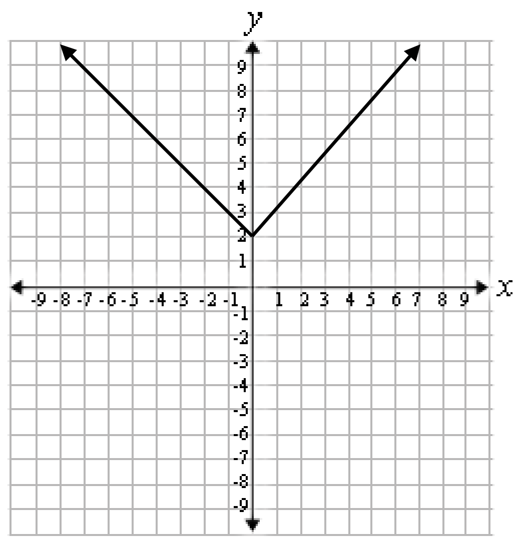
1.

x	y
1	-1
5	3
7	5
25	23

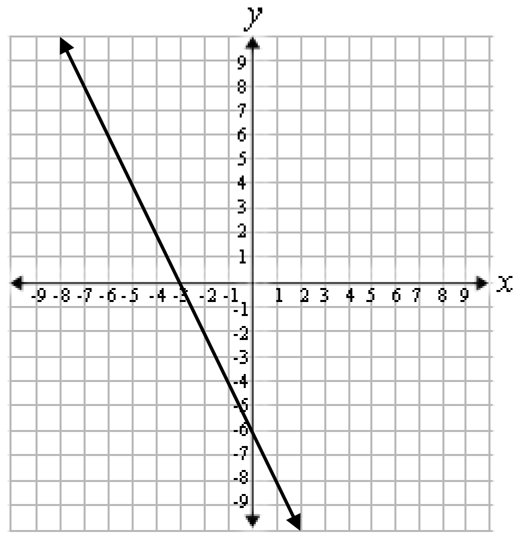
2. $f(x) = 2x + 7$

3. $f(x) = \frac{3}{4}x + 5$

4. $f(x) = |x| + 2$



5. $f(x) = -2x - 6$



Extensions

1. Look through journals, magazines, newspapers, and other printed material to apply the skills you have acquired in learning how to graph and how to understand the graphs you have created.

2. Use graphing techniques when studying data analysis. Look for statistical reports that are in print or on various media such as television or the Internet to explain the meanings of the graphical representation of the statistical reports.

3. This website helps you to explore how the graph of a linear function changes as the slope and y-intercept change.
   http://standards.nctm.org/document/eexamples/chap7/7.5/index.htm

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards, p. 296-318
2008 The Final Report of the National Mathematics Advisory Panel, p. 16, 29
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools, p. 9-12

Graphing Linear Functions – Slope-Intercept
An ADE Mathematics Lesson
Days 16-20

Author	ADE Content Specialists
Grade Level	9th grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 2: Functions and Relationships
PO 1. Sketch and interpret a graph that models a given context, make connections between the graph and the context, and solve maximum and minimum problems using the graph.
PO 4. Use equations, graphs, tables, descriptions, or sets of ordered pairs to express a relationship between two variables.
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.
PO 3. Write an equation given a table of values, two points on the line, the slope and a point on the line, or the graph of the line.
PO 4. Determine from two linear equations whether the lines are parallel, perpendicular, coincident, or intersecting but not perpendicular.
Concept 4: Analysis of Change
PO 1. Determine the slope and intercepts of the graph of a linear function, interpreting slope as a constant rate of change.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO 6. Describe how changing the parameters of a linear function affect the shape and position of its graph.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).

Connects To

Mathematics HS:
Strand 2: Data Analysis, Probability, and Discrete Mathematics
Concept 1: Data Analysis (Statistics)
PO 2. Organize collected data into an appropriate graphical representation with or without technology.

Strand 3: Patterns, Algebra, and Functions
Concept 1: Patterns
PO 1. Recognize, describe, and analyze sequences using tables, graphs, words, or symbols; use sequences in modeling.

Strand 4: Geometry and Measurement
Strand 2: Coordinate Geometry
PO 1. Determine how to find the midpoint between two points in the coordinate plane.
PO 3. Determine the distance between two points in the coordinate plane.
PO 4. Verify characteristics of a given geometric figure using coordinate formulas for distance, midpoint, and slope to confirm parallelism, perpendicularity, and congruence.
Concept 4: Measurement
PO 2. Determine the new coordinates of a point when a single transformation is performed on a 2-dimensional figure.
PO 3. Sketch and describe the properties of a 2-dimensional figure that is the result of two or more transformations.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 5. Summarize and communicate mathematical ideas using formal and informal reasoning.

Aligns To	Connects To
Mathematics HS: **Strand 5: Structure and Logic** **Concept 2: Logic, Reasoning, Problem Solving, and Proof** **PO 3.** Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem. **PO 6.** Synthesize mathematical information from multiple sources to draw a conclusion, make inferences based on mathematical information, evaluate the conclusions of others, analyze a mathematical argument, and recognize flaws or gaps in reasoning.	

Overview

Graphing linear functions is very important. Representing linear functions in several different ways builds conceptual understanding of graphing.

Purpose

This lesson emphasizes graphing linear functions by using a table of values, plotting the x- and y-intercept or by using the slope-intercept form of the equation of a line.

Materials

- Graphing worksheets
- Graph paper
- Ruler

Objectives

Students will:

- Use equations, graphs, tables, descriptions, or sets of ordered pairs to express a relationship between two variables.
- Graph a linear equation in two variables.
- Determine the slope and intercepts of the graph of a linear function, interpreting slope as a constant rate of change.
- Describe how changing the parameters of a linear function affect the shape and position of its graph.
- Write an equation given a table of values, two points on the line, the slope and a point on the line, or the graph of the line.
- Sketch and interpret a graph that models a given context, make connections between the graph and the context.

Lesson Components

Prerequisite Skills: The lesson requires the basics of graphing in a coordinate plane. This lesson builds on the grade 6 skill of graphing ordered pairs in any quadrant of the coordinate plane. This lesson builds on the grade 7 skills of using a table of values to graph an equation or proportional relationship and to describe the graph's characteristics. This lesson builds on grade 8 skill of determining if a relationship represented by a graph or table is a function.

Vocabulary: *Cartesian coordinate system, coordinate plane, graph, ordered pairs, origin, x-coordinate, y-coordinate, quadrant, axes, element, domain, range, independent variable set, dependent variable set, rule of correspondence, x-intercept, y-intercept, slope, rate of change, relation, function, linear function, vertical line, horizontal line, slope-intercept form of the equation of a line, rise/run, parallel lines, perpendicular lines, coincident lines, intersecting lines.*

Session 1 (2 days)

1. Graph linear functions by using the slope-intercept form of the equation of a line.
2. Determine the slope and intercepts of the graph of a linear function, interpreting slope as a constant rate of change.

Session 2 (2 days)

1. Determine the slope and intercepts of the graph of a linear function, interpreting slope as a constant rate of change.
2. Discover how changing the parameters of a linear function change the shape of the graph.
3. Sketch and interpret a graph that models a given context, make connections between the graph and the content.

Session 3 (1 day)

1. Determine from two linear equations whether the lines are parallel, perpendicular, coincident, or intersecting but not perpendicular.

Assessment

There are three assessments, one after each session, which will help pinpoint misconceptions before moving on to more complex comparisons.

Session 1 – Graphing Linear Functions – Slope-Intercept

There are many different ways to graph linear functions. We have learned how to graph them by using a table of values and by using the x- and y-intercepts. A third way that we can graph an equation is by placing it in slope-intercept form $f(x) = mx + b$. The *m* represents the slope and the *b* is the y-intercept. We first draw a point where the line is going to intercept the y-axis. From this point, we use $\frac{rise}{run}$ to determine a second point. If the run is positive, we move to the right from the y-intercept. If it is negative, we move to the left. If the rise is positive, we move up and if it's negative, we move down. You may do the rise first or the run first and then do the other from where the new point would be. If an equation is not already in slope-intercept form, solve it so that it is in that form.

To graph an equation using slope-intercept form follow this process:

- Place the equation in slope intercept form.
- Graph the y-intercept.
- Using rise over run, find the 2nd point.
- Draw a line connecting the points using the representation of an infinite line.

Remember all the suggestions for making a well-drawn graph. Fundamentals of Graphing Linear Function reviewed these suggestions and they are repeated here.

When graphing, there are some **expectations** to which we need to adhere. These include:

1. Use graph paper and graph using a pencil.
2. Use a straightedge to make sure that your lines are straight for linear equations.
3. The units on the axes need to be at equal intervals and the units need to be marked so that others reading your graph will know what each tick mark represents on both the x- and y-axes. The units on the x-axis do not need to be the same as the units on the y-axis.
4. The x- and y-axes represent infinite lines so place arrows on the end of each axis.
5. The graph of a linear function is a line not a line segment. Place arrows at the end of the line to indicate the infinite length.
6. Make sure your graph crosses both axes when you extend the line.
7. If you are graphing points, make sure to indicate the points by making them large enough to be visible to the reader. When working with points, it's a good idea to list the coordinates of the point near the graph of the point.
8. Mark the equation of the line on or near the line.
9. If you graph the equation correctly, it should look exactly the same as another person graphing the same equation.
10. Do not graph more than one equation on a set of axes unless the problem asks you to do so. If you are graphing more than one line on the same set of axes, color code or use a legend to indicate which line is the graph of which equation.
11. If you are sketching a graph after having graphed an equation on a graphing calculator, indicate what window you used and the scale.

Example 1:

Graph $f(x) = 2x + 4$ using slope-intercept.

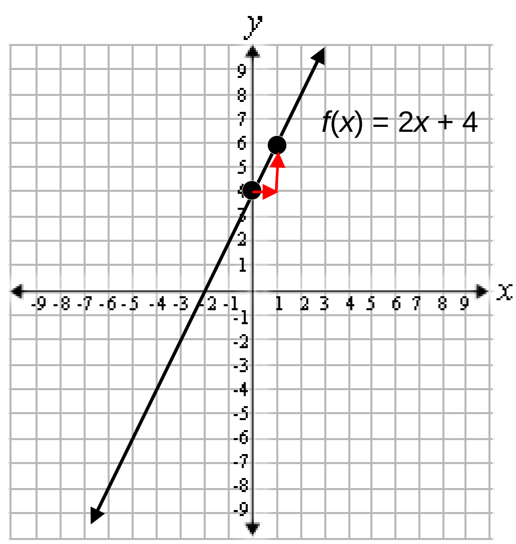
Solution:

Remember that the proper form for slope-intercept equation of a line is $f(x) = mx + b$ where m is the slope and b is the y-intercept.

$$\begin{array}{c} f(x) = 2x + 4 \\ \uparrow \quad \uparrow \\ f(x) = mx + b \end{array}$$

In this case, the slope is 2 which is the same as $\frac{2}{1}$ and the y-intercept is 4.

Begin at 4 on the y-axis. Remember that $\frac{2}{1} = \frac{rise}{run}$. Since the run is positive, move 1 unit to the right from the 4 on the y-axis. Since the rise is positive, move 2 units up from the run. Draw a line connecting the points (0, 4) and (1, 6).



Example 2:

Graph $f(x) = -3x + 2$ using slope-intercept.

Solution:

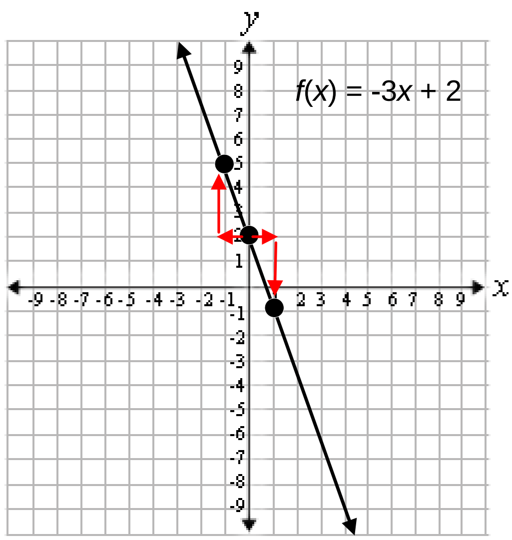
Remember that the proper form for slope-intercept equation of a line is $f(x) = mx + b$ where m is the slope and b is the y-intercept.

$$f(x) = -3x + 2$$

$$f(x) = mx + b$$

In this case, the slope is -3, which is the same as $\frac{-3}{1}$ and the y-intercept is 2.

Begin at 2 on the y-axis. Remember that $\frac{-3}{1} = \frac{rise}{run}$. Since the run is positive, move 1 unit to the right from the 2 on the y-axis. Since the rise is negative, move 3 units down from the run. Draw a line connecting the points (0, 2) and (1, -1). Note that since $\frac{3}{-1} = \frac{-3}{1}$, it is also possible to move one unit to the left of 2 and then move 3 units up from that point. That point is (-1, 5) and is on the graph of the linear function as well.



Example 3:

Graph $f(x)=\frac{2}{5}x-4$ using slope-intercept.

Solution:

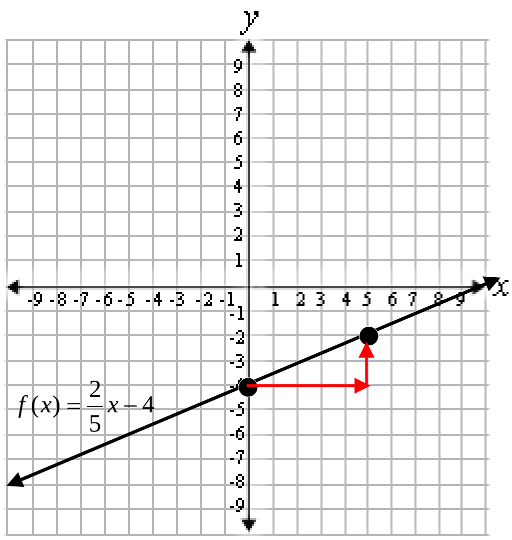
Remember that the proper form for slope-intercept equation of a line is $f(x)=mx+b$ where m is the slope and b is the y-intercept.

$$f(x)=\frac{2}{5}x-4$$

$$f(x)=mx+b$$

In this case, the slope is $\frac{2}{5}$ and the y-intercept is -4.

Begin at -4 on the y-axis. Remember that $\frac{2}{5}=\frac{rise}{run}$. Since the run is positive, move 5 units to the right from the -4 on the y-axis. Since the rise is positive, move 2 units up from the run. Draw a line connecting the points (0, -4) and (5, -2).



Example 4:

Graph $f(x) = -6$ using slope-intercept.

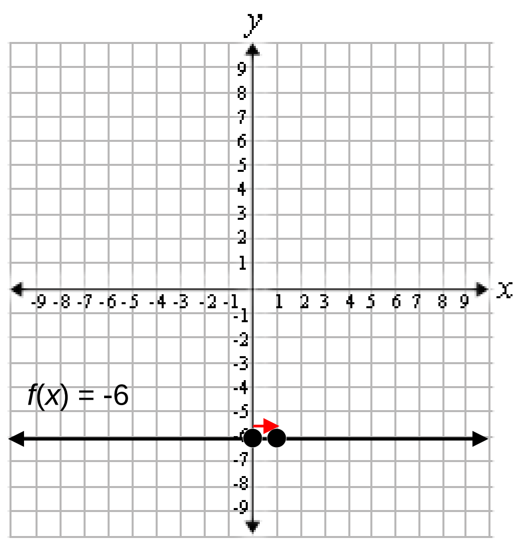
Solution:

Remember that the proper form for slope-intercept equation of a line is $f(x) = mx + b$ where m is the slope and b is the y-intercept.

$$\begin{array}{c} f(x) = 0x - 6 \\ \uparrow \quad \uparrow \\ f(x) = mx + b \end{array}$$

In this case, the slope is 0 and the y-intercept is --6.

Begin at --6 on the y-axis. Remember that $\frac{0}{1} = \frac{rise}{run}$. Since the run is positive, move 1 unit to the right from the -6 on the y-axis. Since the rise is 0, move no units up from the run. Draw a line connecting the points (0, -6) and (1, -6).



Example 5:

Graph $f(x)+2=3-x$ using slope-intercept.

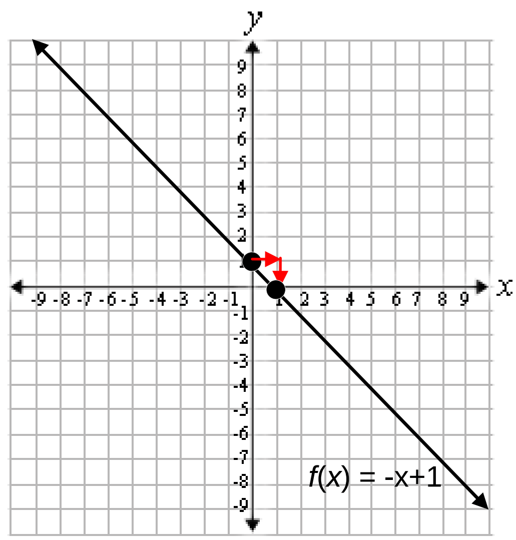
Solution:

First place the equation in slope-intercept form.

$$f(x)+2=3-x \Rightarrow f(x)+2=-x+3 \Rightarrow f(x)=-x+3-2 \Rightarrow f(x)=-x+1$$

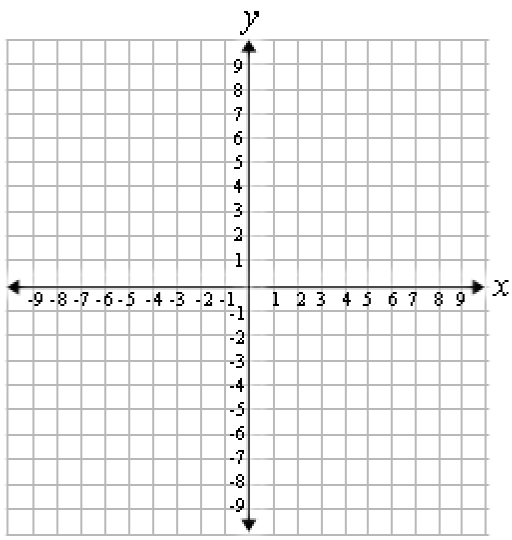
Remember that the proper form for slope-intercept equation of a line is $f(x)=mx+b$ where m is the slope and b is the y-intercept. In this case, the slope is $-1=\frac{-1}{1}$ and the y-intercept is 1.

Begin at 1 on the y-axis. Remember that $\frac{-1}{1}=\frac{rise}{run}$. Since the run is positive, move 1 unit to the right from 1 on the y-axis. Since the rise is negative, move 1 unit down from the run. Draw a line connecting the points (0, 1) and (1, 0).

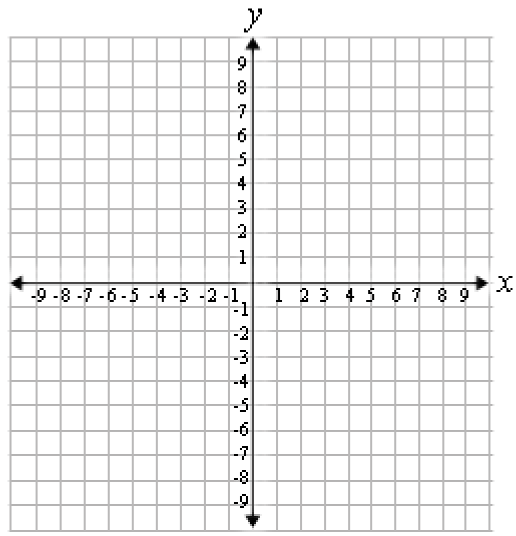


For each of the following problems, graph the linear function using slope-intercept. In the work space that follows each graph, explain how you created the graph. Be very specific.

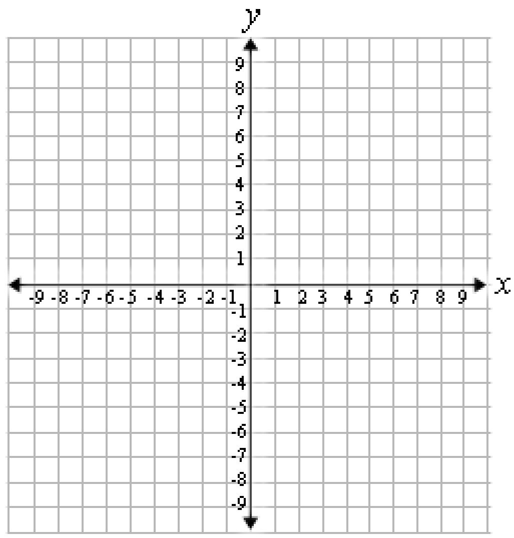
1. $f(x) = 3x - 6$



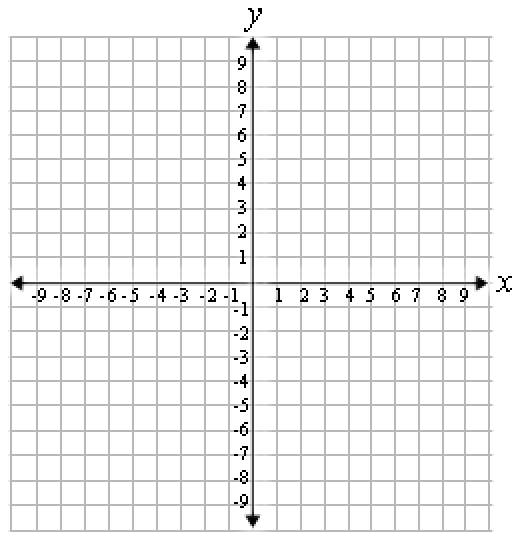
2. $f(x) = -2x - 3$



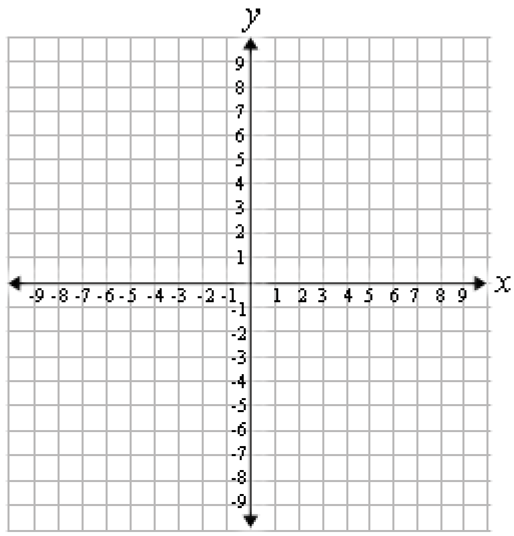
3. $f(x) = \frac{3}{4}x + 2$



4. $f(x) = 3$



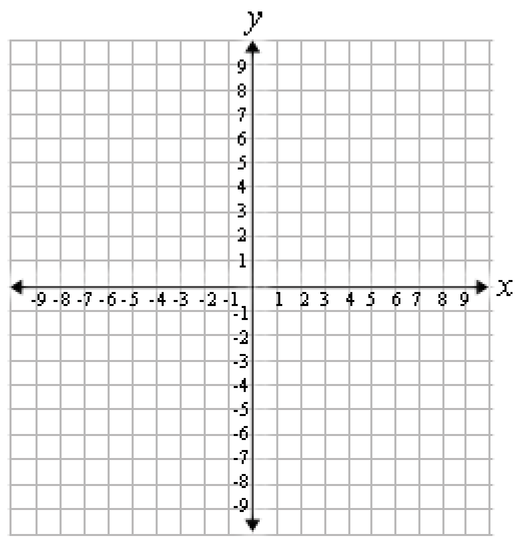
5. $f(x) - 3 = 1 - 2x$



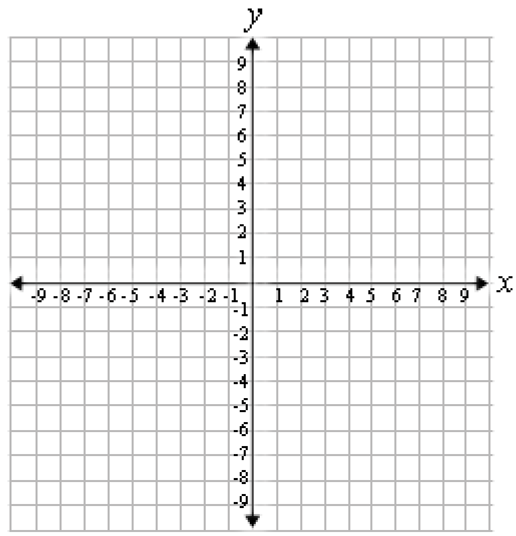
Graphing Linear Functions – Slope-Intercept
Assessment 1

For each of the following problems, graph the linear function using slope-intercept. In the work space that follows each graph, explain how you created the graph. Be very specific.

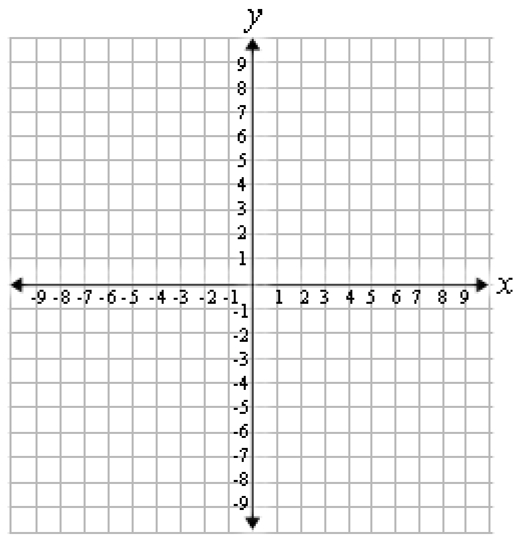
1. $f(x) = 2x + 4$



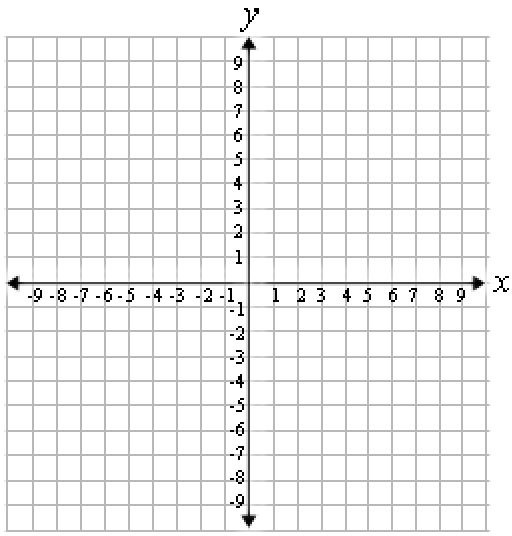
2. $f(x) = -2x + 3$



3. $f(x) = \frac{2}{3}x + 3$



4. $f(x) = -3$



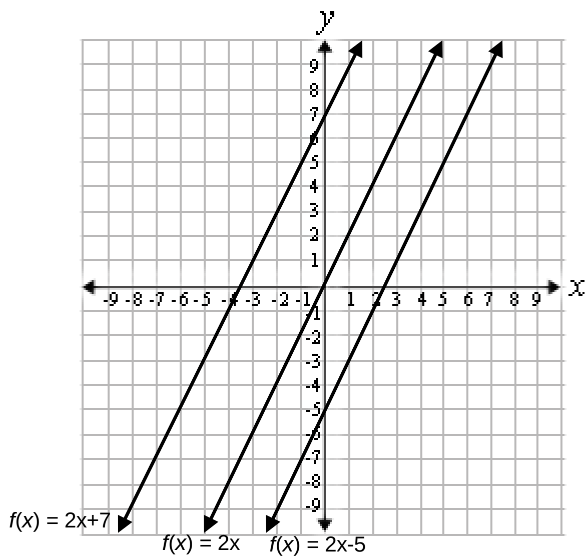
5. In your own words, explain how to graph a function using the slope and y-intercept of that function.

Session 2 – Graphing Linear Functions – Slope-Intercept

Consider what happens to the graph of $f(x) = 2x + 4$ when we change only one parameter. If we change the y-intercept only, the slope will be exactly the same. The difference in the shape of the graph will be where the line intersects the y-axis.

Study the graphs of the functions when the y-intercept is changed:

$f(x) = 2x + 7$
$f(x) = 2x$
$f(x) = 2x - 5$



What happened to the line when the y-intercept changed?

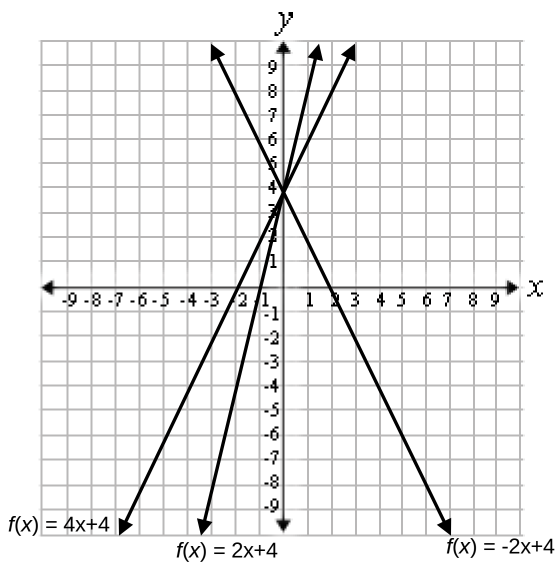
What is the relationship of the graphs of the lines to each other?

Summarize what changing the y-intercept does for a linear function.

Consider what happens to the graph of $f(x) = 2x + 4$ when we change only one parameter. If we change the slope only, the y-intercept stays the same. The difference in the shape of the graph will be in its steepness.

Study the graphs of the function when the slope is changed:

$f(x) = 2x + 4$
$f(x) = 4x + 4$
$f(x) = -2x + 4$

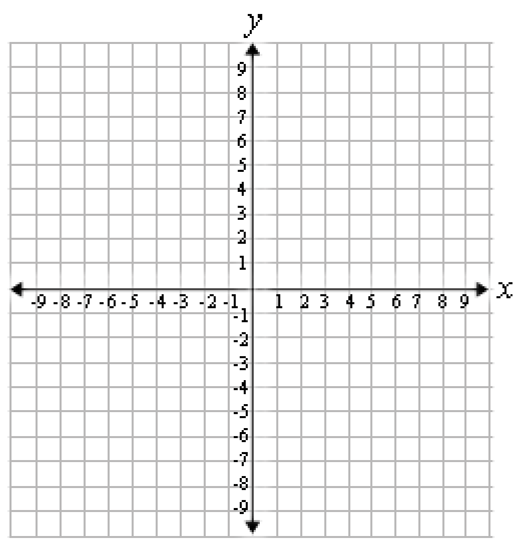


What happened to the line when the slope changed?

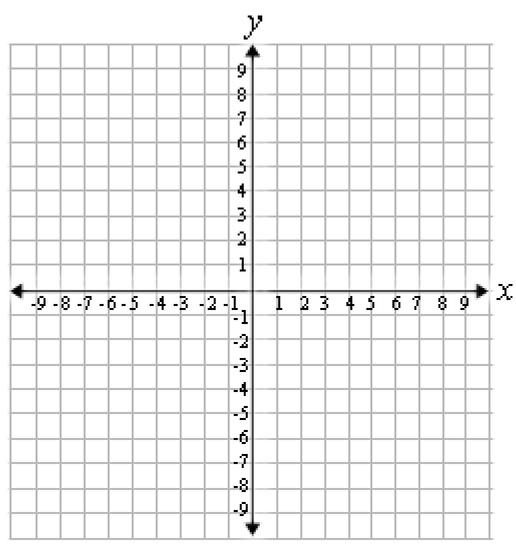
What is the relationship of the graphs of the lines to each other?

Summarize what changing the slope does for a linear function.

In the space below, make up a linear function and graph it. Then change its y-intercept two times and graph each of these new functions. Describe the changes in the graphs of these lines when the y-intercept is changed.



In the space below, make up a linear function and graph it. Then change its slope two times and graph each of these new functions. Describe the changes in the graphs of these lines when the slope is changed.



Many times a graph is used to represent a contextual problem. Study the following problems and the graphs that follow them.

Example 1:

A phone company has many different payment plans for a cell phone. One of them charges a flat rate of $10 per month for the first 200 minutes of use and then 5 cents for each minute after the original 200 minutes. Represent the cost for this cell phone plan graphically.

Solution:

First, define a function that represents the information in the problem.

Let x = the number of minutes used over 200 minutes.

Let f(x) = the monthly cost of the plan.

Then, $f(x) = .05x + 10$

Let's make a table of values.

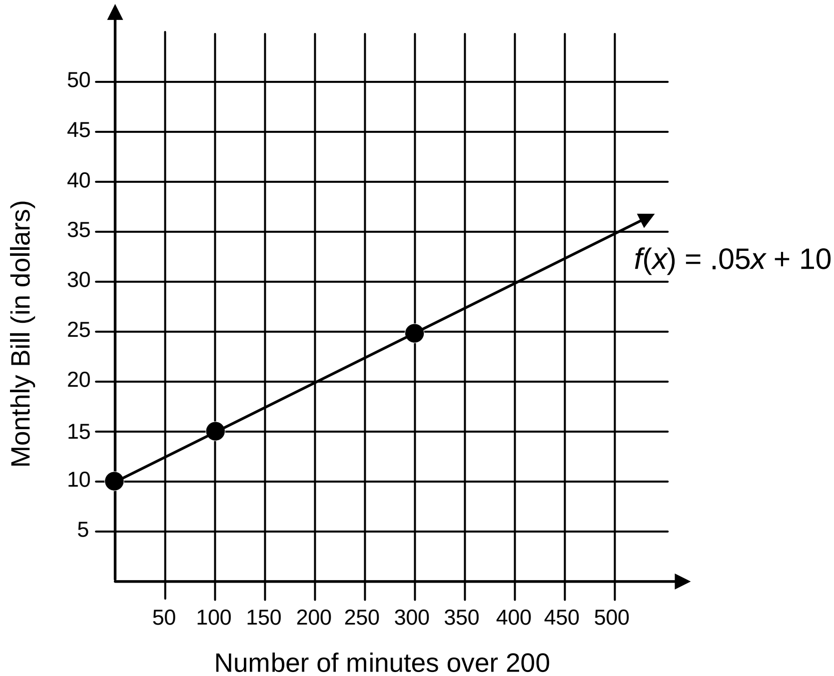
If the customer uses only 200 minutes for the month, the cost is $10.

If the customer uses 300 minutes for the month, the cost increases to $10 plus (300-200) minutes • $.05. This equals $10 + 100 • $.05 which is $15.

If the customer uses 500 minutes for the month, the cost increases to $10 plus (500-200) minutes • $.05. This equals $10 + 300 • $.05 which is $25.

x	**f(x)**
0	$10
100	$15
300	$25

Now let's graph these values to find the line of our function



Answer the following questions about the graph.

1. **Why is it important to indicate the function before graphing the information in the problem?**

2. **List some advantages for graphing the function as described by the information in the problem.**

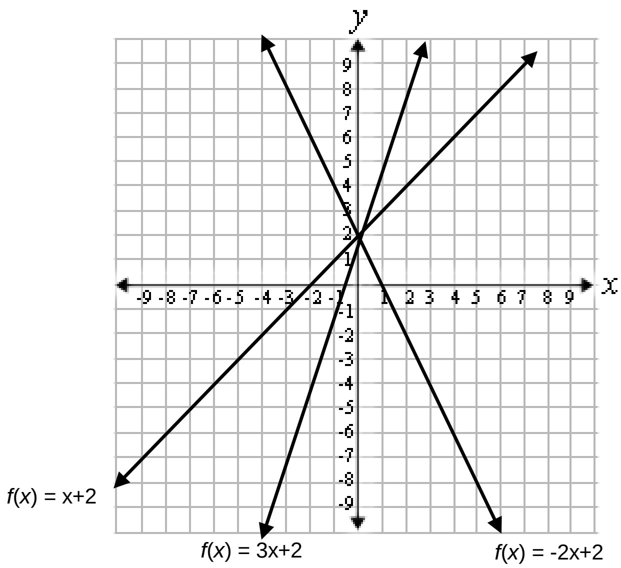
Graphing Linear Functions – Slope-Intercept
Assessment 2

1. Study the graphs of the function when the slope is changed:

$f(x) = 3x + 2$

$f(x) = x + 2$

$f(x) = -2x + 2$



What happened to the line when the slope changed?

What is the relationship of the graphs of the lines to each other?

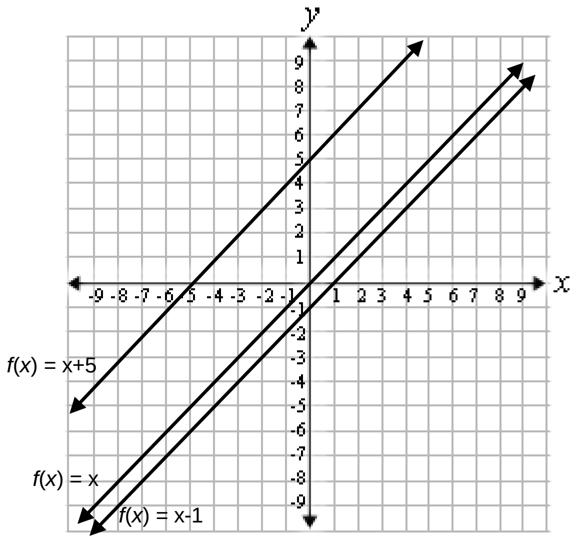
Summarize what changing the slope does for a linear function.

2. Study the graphs of the functions when the y-intercept is changed:

$f(x) = x - 1$

$f(x) = x$

$f(x) = x + 5$



What happened to the line when the y-intercept changed?

What is the relationship of the graphs of the lines to each other?

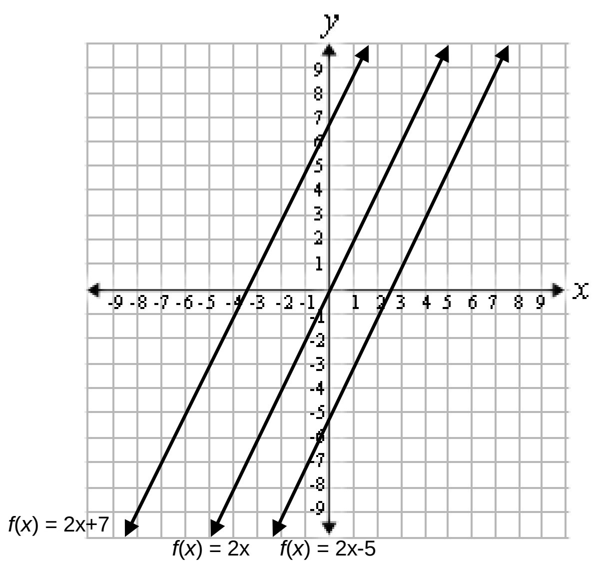
Summarize what changing the y-intercept does for a linear function.

Session 3 – Graphing Linear Functions – Slope-Intercept

It is possible to determine whether two lines are parallel, perpendicular, intersecting, or coincident by looking at the slopes of the lines. Recall the graphs and what they looked like when we changed only the y-intercept. The lines were parallel.

Parallel lines have the same slope.

$f(x) = 2x + 7$
$f(x) = 2x$
$f(x) = 2x - 5$



In each function, the slope of the line is 2. All three lines are parallel.

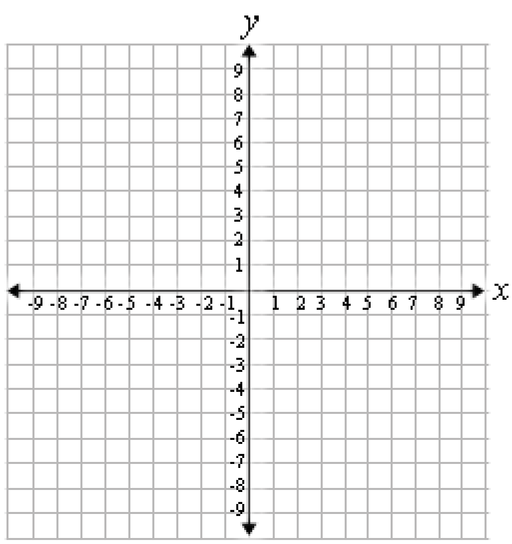
Example 2:

Consider the set of functions:

$f(x) = x + 5$
$f(x) = x$
$f(x) = x - 5$

What do you know about this set of functions? What will the graphs of the functions be? How are the three functions related?

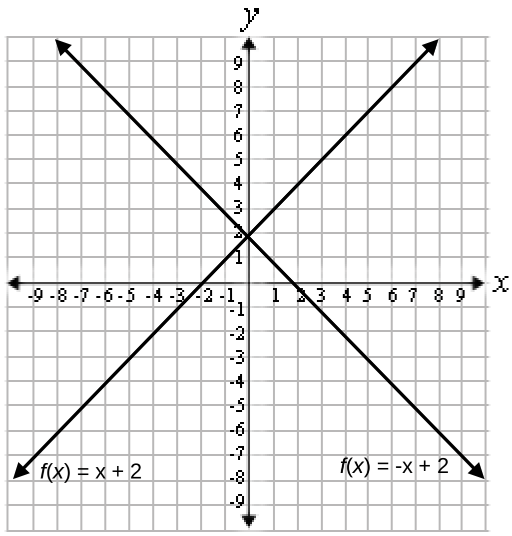
Graph the three functions on the same set of axes to check your answers. Use any method you wish to graph the functions (table of values, intercepts, or slope-intercept) but show your work.



Now, let's look at the graphs of some other functions.

Example 3:

Study the following graph of two functions.



Both graphs have a y-intercept of 2. If we use $\frac{rise}{run}$, the graph that is decreasing has a slope of $\frac{-1}{1}$ while the graph that is increasing has a slope of $\frac{1}{1}$.

The equations of the functions then become

$$f(x) = x + 2$$
$$f(x) = -x + 2$$

Label each graph with its proper equation. When we examine the lines we see that they are perpendicular to each other. The slopes are -1 and 1 which are negative reciprocals of each other.

Parallel lines have the same slope.

The slopes of perpendicular lines are negative reciprocals of each other.

Example 4:

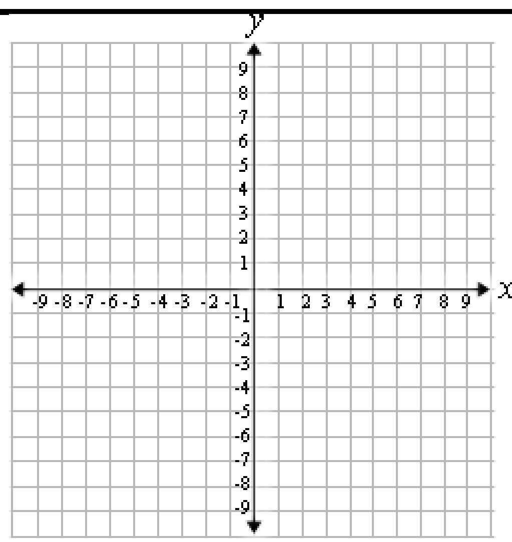
Graph the functions below and determine their relationship to each other.

$$f(x) = \frac{2}{3}x + 3$$

$$f(x) = -\frac{3}{2} - 4$$

Let's look at each function separately. The slope of $f(x) = \frac{2}{3}x + 3$ is $\frac{2}{3}$ and its y-intercept is 3.

The slope of $f(x) = -\frac{3}{2}x - 4$ is $-\frac{3}{2}$ and its y-intercept is -4. Graph each line showing all your work and determine the relationship of each function to the other. Justify your answer.



Lines can also be coincident which means that they are exactly the same line and every point on one line is exactly the same as every point on the other line. If lines are not coincident or parallel, they are intersecting. Intersecting lines do not have to have to be perpendicular. They have one point in common.

Consider the following sets of functions. Determine if they are parallel, perpendicular, coincident, or intersecting and not perpendicular. Justify your answer.

1. $$f(x) = 2x$$
   $$f(x) = -2x$$

2. $$f(x) = 3x + 2$$
   $$f(x) = 3x - 2$$

3. $$f(x) = \frac{x}{2} + 5$$
   $$f(x) = -2x + 3$$

4. $$f(x) - 2 = x + 3$$
   $$f(x) - 3 = x + 2$$

5. $f(x) = 6$
$x = 2$

6. $f(x) = 3x + 2$
$f(x) = -2x + 2$

7. Summarize the relationships two lines can have in relationship to each other.

Graphing Linear Functions – Slope-Intercept
Assessment 3

Consider the following sets of functions. Determine if they are parallel, perpendicular, coincident, or intersecting and not perpendicular. Justify your answer.

1. $f(x) = 3x$
   $f(x) = -3x$

2. $f(x) = 4x + 1$
   $f(x) = 4x - 1$

3. $f(x) = \frac{x}{3} + 1$
   $f(x) = -3x + 4$

4. $$f(x) - 3 = 2x + 3$$
$$f(x) - 2 = 2x + 4$$

5. $$f(x) = 3$$
$$x = -3$$

Extensions

1. Graph functions using a graphing calculator.

Graphing Linear Functions – Slope-Intercept Extension

We can graph an equation using a graphing calculator. We need to put the equation in "$y =$" after you have placed the equation in slope-intercept form. Check to make sure your window is correct for the equation you are graphing. Usually, we use a standard window in which the minimum value for x is –10, the maximum value for x is 10, the minimum value for y is –10 and the maximum value for y is 10. We usually use a scale of one which means that each unit between –10 and 10 is 1. If the scale were 2, then each unit between –10 and 10 would represent 2. After putting in the equation and checking to make sure that the window is correct, hit the "graph" key and the equation will show up on the graphing screen. Make sure to be accurate when sketching the graph on your paper.

Try putting some of the equations you graphed in this lesson into your graphing calculator to see if you obtained the correct shapes of the graphs.

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards
2008 The Final Report of the National Mathematics Advisory Panel
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools

Simplifying Algebraic Expressions
An ADE Mathematics Lesson
Days 21-27

Author	ADE Content Specialists
Grade Level	9^{th} grade
Duration	Seven days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.
PO 8. Simplify and evaluate polynomials, rational expressions, expressions containing absolute value, and radicals.
PO 9. Multiply and divide monomial expressions with integer exponents.
PO 10. Add, subtract, and multiply polynomial and rational expressions.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.
PO 4. Generalize a solution strategy for a single problem to a class of related problems; explain the role of generalizations in inductive and deductive reasoning.
PO 7. Find structural similarities within different algebraic expressions and geometric figures.

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 2: Numerical Operations
PO 1. Solve word problems involving absolute value, powers, roots, and scientific notation.
PO 2. Summarize the properties of and connections between real number operations; justify manipulations of expressions using the properties of real number operations.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 12. Factor quadratic polynomials in the form of ax^2+bx+c where a, b, and c are integers.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.

Overview

Simplifying algebraic expressions is an important skill. There are several types of problems that will be explored in this lesson. The first steps are to represent algebraic expressions in equivalent forms and simplify rational expressions. Next, you will explore multiplication and division of monomial expressions with whole number exponents. Lastly, you will solve problems with addition, subtraction, and multiplication of polynomial and rational expressions.

Purpose

Simplifying algebraic expressions allows you to work with more complicated algebraic equations and concepts. Adding, subtracting, multiplying, and dividing monomial and polynomial expressions allows you to simplify work before moving on to more complicated mathematical tasks.

Materials

- Algebraic worksheets

Objectives

Students will:

- Simplify algebraic expressions with whole number exponents.
- Simplify rational expressions.
- Multiply and divide monomial expressions with whole number exponents.
- Add, subtract, and multiply polynomial and rational expressions.

Lesson Components

Prerequisite Skills: This lesson builds on grade 7 and 8 skills of simplifying and solving equations as well as simplifying numeric expressions with positive exponents.
You must be able to factor trinomials in order to complete Session 2.

Vocabulary: *monomial, binomial, polynomial, exponent, zero exponent, rational expression, leading coefficient, degree of polynomial*

Session 1: Multiplying Monomials, Dividing Monomials, and Zero Exponents (4 days)

1. Simplify algebraic expressions with whole number expressions.
2. Simplify rational expressions.
3. Multiply and divide monomial expressions with whole number exponents.

Session 2 (3 days)

1. Add, subtract, and multiply polynomial and rational expressions.

Assessment

There are two assessments that will help pinpoint misconceptions before moving on to more complex algebraic expressions and equations.

Simplifying Algebraic Expressions
Background Vocabulary

There are many terms with which you have already become familiar. Before moving on to simplifying algebraic expressions and working with rational expressions, let’s review some of these terms. In the space provided define each term in your own words. Afterwards, compare your answer to the definitions provided. Are you saying the same thing? It is beneficial to understand terms in your own words as you gain a deeper understanding of the concept.

Term	**Definition**
algebraic expression	
monomial	
binomial	
polynomial	
degree of a polynomial	
exponent	
leading coefficient	
rational expression	

Term	Definition
algebraic expression	a group of numbers, symbols, and variables that express a single or series of mathematical operations (e.g., $2x + 4 - 16y$)
monomial	an algebraic expression consisting of a single term that does not require any addition or subtraction (e.g., 5y)
binomial	an algebraic expression consisting of two terms (e.g., $x + 3$, $4a - 6$)
polynomial	an expression containing more than one monomial connected by addition or subtraction(e.g., $3x^2 + 2x + 7$, $4x^5 - 9x^3 + 2x + 7$)
degree of a polynomial	the degree of the highest term of the polynomial (e.g., The degree of $3x + 2x^2 + 4 - 7x^5 - 3x + 10x^4$ is 5 because 5 is the greatest exponent)
exponent	a number placed to the right and above (superscript) a non-zero base that indicates the operation of repeated multiplication (e.g., in 5^7 the exponent is 7)
leading coefficient	the coefficient of the term of the highest degree in a polynomial (e.g., in the expression $15x - 10x^3 - 11x^6 + 7x^2 + 3.5$ -11 is the leading coefficient)
rational expression	the quotient of two polynomials in the form $\frac{A}{B}$, where A and B are polynomials and where *B* can **never** equal 0.(e.g., $\frac{2x+1}{3x^2-9}$, $3x^2 - 9 \neq 0$)

If you are unfamiliar with any of these terms, review them before proceeding with this lesson. It may be helpful to write more examples for each word as part of your review. It is important that you understand these terms in order to successfully complete this lesson.

Simplifying Algebraic Expressions
Session 1 Part 1– Multiplying Monomials

Example 1: Study the following examples and then answer the questions that follow.

$x^2 = x \bullet x$

$y^2 = y \bullet y$

$x^2 y^2 = x \bullet x \bullet y \bullet y$

$(xy)^2 = (xy)(xy)$
$= x \bullet x \bullet y \bullet y$
$= x^2 \bullet y^2$

$(x^2 y)^2 = (x^2 y)(x^2 y)$
$= (x \bullet x \bullet y)(x \bullet x \bullet y)$
$= x^4 y^2$

$(x^2 y^2)^2 = (x^2 y^2)(x^2 y^2)$
$= (x \bullet x \bullet y \bullet y)(x \bullet x \bullet y \bullet y)$
$= (x \bullet x \bullet x \bullet x)(y \bullet y \bullet y \bullet y)$
$= x^4 y^4$

$(x^2 y^2)^3 = (x^2 y^2)(x^2 y^2)(x^2 y^2)$
$= (x \bullet x \bullet y \bullet y)(x \bullet x \bullet y \bullet y)(x \bullet x \bullet y \bullet y)$
$= (x \bullet x \bullet x \bullet x \bullet x \bullet x)(y \bullet y \bullet y \bullet y \bullet y \bullet y)$
$= x^6 y^6$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Example 2: Study the following examples and then answer the questions that follow.

$$2x^2 = 2 \bullet x \bullet x$$

$$3x^2 = 3 \bullet x \bullet x$$

$$(2x^2)(3x^3) = (2 \bullet x \bullet x)(3 \bullet x \bullet x \bullet x)$$
$$= 2 \bullet 3(x \bullet x \bullet x \bullet x \bullet x)$$
$$= 6x^5$$

$$(2x^2 \bullet 3x^3)^2 = (2x^2 \bullet 3x^3)(2x^2 3x^3)$$
$$= (2 \bullet x \bullet x \bullet 3 \bullet x \bullet x \bullet x)(2 \bullet x \bullet x \bullet 3 \bullet x \bullet x \bullet x)$$
$$= (2 \bullet 3 \bullet 2 \bullet 3)(x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x)$$
$$= 36x^{10}$$

$$(2x^2 \bullet 3x^3)^3 = (2x^2 \bullet 3x^3)(2x^2 \bullet 3x^3)(2x^2 \bullet 3x^3)$$
$$= (2 \bullet x \bullet x \bullet 3 \bullet x \bullet x \bullet x)(2 \bullet x \bullet x \bullet 3 \bullet x \bullet x \bullet x)(2 \bullet x \bullet x \bullet 3 \bullet x \bullet x \bullet x)$$
$$= (2 \bullet 3 \bullet 2 \bullet 3 \bullet 2 \bullet 3)(x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x \bullet x)$$
$$= 216x^{15}$$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Example 3: Study the following examples and then answer the questions that follow.

$$a^3 = a \bullet a \bullet a$$

$$a^2 = a \bullet a$$

$$a^3 a^2 = a \bullet a \bullet a \bullet a \bullet a = a^5$$

$$(a^4)^3 = a^4 \bullet a^4 \bullet a^4$$
$$= a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a = a^{12}$$

$$(a^3b^2)^2 = (a^3b^2)(a^3b^2)$$
$$= (a \bullet a \bullet a \bullet b \bullet b)(a \bullet a \bullet a \bullet b \bullet b)$$
$$= (a \bullet a \bullet a \bullet a \bullet a \bullet a)(b \bullet b \bullet b \bullet b) = a^6b^4$$

$$(a^3b^2)^3$$
$$= (a^3b^2)(a^3b^2)(a^3b^2)$$
$$= (a \bullet a \bullet a \bullet b \bullet b)(a \bullet a \bullet a \bullet b \bullet b)(a \bullet a \bullet a \bullet b \bullet b)$$
$$= (a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a \bullet a)(b \bullet b \bullet b \bullet b \bullet b \bullet b) = a^9b^6$$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Example 4: Study the following examples and then answer the questions that follow.

$m^2 = m \bullet m$

$m^4 = m \bullet m \bullet m \bullet m$

$(m^2)(m^4) = m \bullet m \bullet m \bullet m \bullet m \bullet m$
$= m^6$

$(m^2n^4)^2$
$= (m^2n^4)(m^2n^4)$
$= (m \bullet m \bullet n \bullet n \bullet n \bullet n)(m \bullet m \bullet n \bullet n \bullet n \bullet n)$
$= (m \bullet m \bullet m \bullet m)(n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n)$
$= m^4n^8$

$(m^2n^4)^3$
$= (m^2n^4)(m^2n^4)(m^2n^4)$
$= (m \bullet m \bullet n \bullet n \bullet n \bullet n)(m \bullet m \bullet n \bullet n \bullet n \bullet n)(m \bullet m \bullet n \bullet n \bullet n \bullet n)$
$= (m \bullet m \bullet m \bullet m \bullet m \bullet m)(n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n \bullet n)$
$= m^6n^{12}$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Was there a common theme to your answers in each of the four examples? As you can tell, the process of simplifying a monomial raised to a power is not difficult but it can be very tedious. There are rules of exponents that show what happens when monomials are raised to a power.

Rule	Example
$a^m \bullet a^n = a^{m+n}$	$a^6 \bullet a^3 = a^{6+3} = a^9$ $5x^3 \bullet 2x^2 = (5 \bullet 2)(x^{3+2}) = 10x^5$
$(a^m)^n = a^{mn}$	$(x^2)^3 = x^{2\bullet 3} = x^6$ $(5^3)^8 = 5^{3\bullet 8} = 5^{24}$
$(a \bullet b)^n = a^n b^n$	$(2n^3)^2 = (2^2)(n^{3\bullet 2}) = 4n^6$

Look through the examples shown previously. Explain in the space provided how applying the rules of exponents would affect simplifying each example.

Simplify the following algebraic expressions showing your work in the space provided.

1. $(x^2 \bullet y^4 \bullet z)^2$

2. $(-4x \bullet 5x^4)$

3. $(2ab^2)^3$

4. $(-3ab^2c^3)^2(abc)^2$

5. $(-2x^2yz^3)^3$

6. $(-mn^2)(m^2n)(m^3n^3)$

7. $(2y^2z)^2(xz)^3(-4x^2y)^2$

Simplifying Algebraic Expressions
Session 1 Part 2 – Dividing Monomials

We can also divide monomial expressions with whole number exponents. Study the examples and then answer the questions that follow.

Example 1: Study the examples and then answer the questions that follow.

$$\frac{x^4}{x^3}$$

$$= \frac{x \bullet x \bullet x \bullet x}{x \bullet x \bullet x}$$

$$= \frac{\cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet x}{\cancel{x} \bullet \cancel{x} \bullet \cancel{x}}$$

$$= x$$

$$\frac{12x^4}{3x^3}$$

$$= \frac{2 \bullet 2 \bullet 3 \bullet x \bullet x \bullet x \bullet x}{3 \bullet x \bullet x \bullet x}$$

$$= \frac{2 \bullet 2 \bullet \cancel{3} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet x}{\cancel{3} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x}}$$

$$= 2 \bullet 2 \bullet x = 4x$$

$$\frac{12x^4y^3}{3x^3y^2}$$

$$= \frac{2 \bullet 2 \bullet 3 \bullet x \bullet x \bullet x \bullet x \bullet y \bullet y \bullet y}{3 \bullet x \bullet x \bullet x \bullet y \bullet y}$$

$$\frac{2 \bullet 2 \bullet \cancel{3} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet x \bullet \cancel{y} \bullet \cancel{y} \bullet y}{\cancel{3} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{y} \bullet \cancel{y}}$$

$$= 2 \bullet 2 \bullet x \bullet y = 4xy$$

$$\frac{8x^3y^4}{2x^5y^3}$$

$$= \frac{2 \bullet 2 \bullet 2 \bullet x \bullet x \bullet x \bullet y \bullet y \bullet y \bullet y}{2 \bullet x \bullet x \bullet x \bullet x \bullet x \bullet y \bullet y \bullet y}$$

$$\frac{2 \bullet 2 \bullet \cancel{2} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet y \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y}}{\cancel{2} \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet x \bullet x \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{y}}$$

$$= \frac{4y}{x^2}$$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Example 2: Study the examples and then answer the questions that follow.

$$\frac{xy^3z^2}{xyz}$$

$$= \frac{x \bullet y \bullet y \bullet y \bullet z \bullet z}{x \bullet y \bullet z}$$

$$= \frac{\cancel{x} \bullet \cancel{y} \bullet y \bullet y \bullet \cancel{z} \bullet z}{\cancel{x} \bullet \cancel{y} \bullet \cancel{z}}$$

$$= y^2z$$

$$\frac{3xy^3z^2}{15xyz}$$

$$= \frac{3 \bullet x \bullet y \bullet y \bullet y \bullet z \bullet z}{3 \bullet 5 \bullet x \bullet y \bullet z}$$

$$= \frac{\cancel{3} \bullet \cancel{x} \bullet y \bullet y \bullet \cancel{y} \bullet \cancel{z} \bullet z}{\cancel{3} \bullet 5 \bullet \cancel{x} \bullet \cancel{y} \bullet \cancel{z}}$$

$$= \frac{y^2z}{5}$$

$$\left(\frac{xy^3z^2}{xyz}\right)^2$$

$$= \left(\frac{xy^3z^2}{xyz}\right) \bullet \left(\frac{xy^3z^2}{xyz}\right)$$

$$= \left(\frac{x \bullet y \bullet y \bullet y \bullet z \bullet z}{x \bullet y \bullet z}\right)\left(\frac{x \bullet y \bullet y \bullet y \bullet z \bullet z}{x \bullet y \bullet z}\right)$$

$$= \frac{x \bullet x \bullet y \bullet y \bullet y \bullet y \bullet y \bullet y \bullet z \bullet z \bullet z \bullet z}{x \bullet x \bullet y \bullet y \bullet z \bullet z}$$

$$= \frac{\cancel{x} \bullet \cancel{x} \bullet y \bullet y \bullet y \bullet y \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{z} \bullet \cancel{z} \bullet z \bullet z}{\cancel{x} \bullet \cancel{x} \bullet \cancel{y} \bullet \cancel{y} \bullet \cancel{z} \bullet \cancel{z}}$$

$$= y \bullet y \bullet y \bullet y \bullet z \bullet z$$

$$= y^4z^2$$

If the same fraction is simplified first, <u>the result is the same</u>.

$$\left(\frac{xy^3z^2}{xyz}\right)^2$$

$$= \left(\frac{\cancel{x} \bullet \cancel{y} \bullet y \bullet y \bullet \cancel{z} \bullet z}{\cancel{x} \bullet \cancel{y} \bullet \cancel{z}}\right)^2$$

$$= \left(y^2z\right)^2$$

$$= (y \bullet y \bullet z)(y \bullet y \bullet z)$$

$$= (y \bullet y \bullet y \bullet y \bullet z \bullet z)$$

$$= y^4z^2$$

Do you see any patterns in the examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Example 3: Study the examples and then answer the questions that follow.

$$\frac{100a^2b^3c^4}{30a^2b^4c^5}$$

$$= \frac{2 \bullet 2 \bullet 5 \bullet 5 \bullet a \bullet a \bullet b \bullet b \bullet b \bullet c \bullet c \bullet c \bullet c}{2 \bullet 3 \bullet 5 \bullet a \bullet a \bullet b \bullet b \bullet b \bullet b \bullet c \bullet c \bullet c \bullet c \bullet c}$$

$$= \frac{\cancel{2} \bullet 2 \bullet \cancel{5} \bullet 5 \bullet \cancel{a} \bullet \cancel{a} \bullet \cancel{b} \bullet \cancel{b} \bullet \cancel{b} \bullet \cancel{c} \bullet \cancel{c} \bullet \cancel{c} \bullet \cancel{c}}{\cancel{2} \bullet 3 \bullet \cancel{5} \bullet \cancel{a} \bullet \cancel{a} \bullet b \bullet \cancel{b} \bullet \cancel{b} \bullet \cancel{b} \bullet c \bullet \cancel{c} \bullet \cancel{c} \bullet \cancel{c} \bullet \cancel{c}}$$

$$= \frac{2 \bullet 5}{3 \bullet b \bullet c}$$

$$= \frac{10}{3bc}$$

$$\left(\frac{3a^2b^3}{a^3b}\right)^2$$

$$=\left(\frac{3a^2b^3}{a^3b}\right)\left(\frac{3a^2b^3}{a^3b}\right)$$

$$=\left(\frac{3\bullet a\bullet a\bullet b\bullet b\bullet b}{a\bullet a\bullet a\bullet b}\right)\left(\frac{3\bullet a\bullet a\bullet b\bullet b\bullet b}{a\bullet a\bullet a\bullet b}\right)$$

$$=\frac{3\bullet 3\bullet a\bullet a\bullet a\bullet a\bullet b\bullet b\bullet b\bullet b\bullet b\bullet b}{a\bullet a\bullet a\bullet a\bullet a\bullet a\bullet b\bullet b}$$

$$=\frac{3\bullet 3\bullet a\bullet a\bullet a\bullet a\bullet b\bullet b\bullet b\bullet b\bullet b\bullet b}{a\bullet a\bullet a\bullet a\bullet a\bullet a\bullet b\bullet b}$$

$$=\frac{9b^4}{a^2}$$

Simplify the same problem first and obtain the same answer.

$$\left(\frac{3a^2b^3}{a^3b}\right)^2$$

$$=\left(\frac{3\bullet a\bullet a\bullet b\bullet b\bullet b}{a\bullet a\bullet a\bullet b}\right)^2$$

$$=\left(\frac{3\bullet b\bullet b}{a}\right)^2$$

$$=\left(\frac{3\bullet b\bullet b}{a}\right)\left(\frac{3\bullet b\bullet b}{a}\right)$$

$$=\frac{3\bullet 3\bullet b\bullet b\bullet b\bullet b}{a\bullet a}$$

$$=\frac{9b^4}{a^2}$$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions?

Was there a common theme to your answers in each of the examples above? As you can tell, the process of simplifying a rational exponent is not difficult but it can be very tedious. Let's add some new rules to the previous rules of exponents that we discovered.

Rule	**Example**
$a^m \bullet a^n = a^{m+n}$	$a^6 \bullet a^3 = a^{6+3} = a^9$
$(a^m)^n = a^{mn}$	$\left(x^2\right)^3 = x^{2\bullet 3} = x^6$
$(a \bullet b)^n = a^n b^n$	$(2n^3)^2 = 2^2 n^{3\bullet 2}) = 4n^6$
$\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{a^7}{a^3} = a^{7-3} = a^4$
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{x}{3}\right)^3 = \dfrac{x^3}{3^3} = \dfrac{x^3}{27}$

Does it matter when simplifying rational expressions whether the expression is simplified before being raised to a power or after being raised to a power? Justify your answer.

Look through the examples shown previously. Explain in the space provided how applying the rules of exponents would affect simplifying each example.

There may be many ways to simplify an algebraic expression. It is important to understand each step when simplifying an algebraic expression. Which way you choose to simplify an algebraic expression is not as important as understanding the steps as you simplify.

Several previous examples were simplified in more than one way yet we always ended up with the same answer.

Simplify the following algebraic expressions showing your work.

1. $\dfrac{a^5}{a^3}$

2. $\dfrac{4x^8y^5}{x^3y}$

3. $\left(\dfrac{c^2}{4}\right)^2$

4. $\dfrac{6m^3n^2}{16m^5n^6}$

5. $\left(\dfrac{2a^2b^3c}{abc}\right)^2$

6. $\left(\dfrac{10x^2y^2z^2}{25x^3yz}\right)^2$

Simplifying Algebraic Expressions
Session 1 Part 3– Zero Exponents

Example 1: Study the examples and then answer the questions that follow.

$$\frac{x^4}{x^4} = \frac{x \bullet x \bullet x \bullet x}{x \bullet x \bullet x \bullet x} = 1$$

If we apply the rules of exponents that we learned, then we know that $\frac{x^4}{x^4} = x^{4-4} = x^0$.

Since $\frac{x^4}{x^4} = 1$ and $\frac{x^4}{x^4} = x^0$, we learn that $x^0 = 1$.

Similarly,

$\frac{c^3}{c^3} = \frac{c \bullet c \bullet c}{c \bullet c \bullet c} = 1$ and $\frac{c^3}{c^3} = c^{3-3} = c^0$. Therefore $c^0 = 1$.

Similarly,

$\frac{5^2}{5^2} = \frac{5 \bullet 5}{5 \bullet 5} = \frac{1}{1}$ and $\frac{5^2}{5^2} = 5^{2-2} = 5^0$. Therefore $5^0 = 1$

Similarly,

$\frac{a}{a} = 1$ and $\frac{a}{a} = \frac{a^1}{a^1} = a^{1-1} = a^0$. Therefore, $a^0 = 1$

Do you see any patterns in the above examples? If so, what are they?

Explain your answer.

Can you make any generalizations from the examples shown that would help you to simplify monomial expressions when the expression contains a zero exponent?

Was there a common theme to your answers in the example? As you can tell, the process of simplifying a rational exponent is not difficult but it can be very tedious. Let's add some new rules to the previous rules of exponents that we discovered.

Rule	**Example**
$a^m \bullet a^n = a^{m+n}$	$a^6 \bullet a^3 = a^{6+3} = a^9$
$(a^m)^n = a^{mn}$	$\left(x^2\right)^3 = x^{2\bullet 3} = x^6$
$(a \bullet b)^n = a^n b^n$	$(2n^3)^2 = 2^2 n^{3\bullet 2}) = 4n^6$
$\dfrac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$	$\dfrac{a^7}{a^3} = a^{7-3} = a^4$
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \quad (b \neq 0)$	$\left(\dfrac{x}{3}\right)^3 = \dfrac{x^3}{3^3} = \dfrac{x^3}{27}$
$a^0 = 1 \quad (a \neq 0)$	$(3xy)^0 = 1$

Simplify the following algebraic expressions showing your work.

1. $(mn)^0$

2. $\left(\frac{4xy}{5}\right)^0$

3. $\frac{(3abc)^0}{7}$

4. $-4x^0y^0z$

Simplify the following expressions showing your work in the space provided.

1. $(x^2)(y^3)$

2. $(2x^3y)(6x^2y)$

3. $(a^3b^4)(-3a^2b)$

4. $(n^2)^6$

5. $(3x^3)^3$

6. $-5(x^2y^4z^3)^2$

7. $\frac{n^7}{n^3}$

8. $\left(\frac{2xy^2z}{5xy}\right)^2$

9. $\frac{12x^4y^2z^3}{3xyz}$

10. $(xy)^0$

11. $\frac{(a^3b^2c)^0}{2}$

12. $\left(\frac{2cd^4}{12c^2d}\right)^0$

Simplifying Algebraic Expressions
Assessment 1

Match each term to its definition.

_____**1. binomial**	A. an algebraic expression consisting of a single term that does not require any addition or subtraction (e.g., 5y)
_____**2. leading coefficient**	B. the degree of the highest term of the polynomial (e.g., The degree of $3x+2x^2+4-7x^5-3x+10x^4$ is 5 because 5 is the greatest exponent)
_____**3. Monomial**	C. an algebraic expression consisting of two terms (e.g., $x+3$, $4a-6$)
_____**4. Exponent**	D. a group of numbers, symbols, and variables that express a single or series of mathematical operations (e.g., $2x+4-16y$)
_____**5. degree of a polynomial**	E. a number placed to the right and above (superscript) a non-zero base that indicates the operation of repeated multiplication (e.g., in 5^7 the exponent is 7)
_____**6. polynomial**	F. the quotient of two polynomials in the form $\frac{A}{B}$, where A and B are polynomials and where B can **never** equal 0.(e.g., $\frac{2x+1}{3x^2-9}$, $3x^2-9\neq 0$)
_____**7. algebraic expression**	G. an expression containing more than one monomial connected by addition or subtraction(e.g., $3x^2+2x+7,\quad 4x^5-9x^3+2x+7$)
_____**8. rational expression**	H. the coefficient of the term of the highest degree in a polynomial (e.g., in the expression $15x-10x^3-11x^6+7x^2+3.5$ -11 is the leading coefficient)

Simplify the following algebraic expressions showing your work.

9. $\dfrac{a^7}{a^4}$

10. $\dfrac{6x^3y3}{2x^2y}$

11. $\left(\dfrac{b^3}{3}\right)^2$

12. $\dfrac{4m^2n^3}{18m^4n^5}$

13. $\left(\frac{-3a^3b^3c^2}{a^2bc}\right)^2$

14. $\left(\frac{5x^2y^2z^3}{20x^4yz^2}\right)^2$

15. $(ab)^0$

16. $\dfrac{(x^3y^2z)^0}{-5}$

17. $\left(\dfrac{7mn^4}{21m^2n}\right)^0$

18. $\left(\dfrac{3}{xy^4}\right)^0$

Simplifying Algebraic Expressions
Session 2 – Operations with Rational Expressions
***Note: You must be able to factor trinomials in order to complete this session.**

We can add, subtract, multiply, and divide rational expressions.

Example 1: Study the following problems which add fractions with like denominators.

$$\frac{2}{9}+\frac{5}{9}$$
$$=\frac{2+5}{9}$$
$$=\frac{7}{9}$$

$$\frac{1}{4}+\frac{1}{4}$$
$$=\frac{1+1}{4}$$
$$=\frac{2}{4}$$

This can be simplified to $\frac{1}{2}$

$$\frac{5}{12}+\frac{3}{12}$$
$$=\frac{5+3}{12}$$
$$=\frac{8}{12}$$

This can be simplified to $\frac{2}{3}$

$$-\frac{3}{5}+\frac{4}{5}$$
$$=\frac{-3+4}{5}$$
$$=\frac{1}{5}$$

In the work space provided, write the process used to add fractions with like denominators in your own words.

Example 2: Study the following problems which subtract fractions with like denominators.

$$\frac{3}{4}-\frac{2}{4}=\frac{3-2}{4}$$
$$=\frac{1}{4}$$

$$\frac{8}{9}-\frac{5}{9}=\frac{8-5}{9}$$
$$=\frac{3}{9}$$
$$=\frac{1}{3}$$

$$\frac{5}{12}-\frac{2}{12}=\frac{5-2}{12}$$
$$=\frac{3}{12}$$
$$=\frac{1}{4}$$

$$\frac{1}{3}-\left(-\frac{1}{3}\right)=\frac{1}{3}-\frac{-1}{3}$$

*Remember that a negative fraction means *either* the numerator is negative OR the denominator is negative, but BOTH are not negative.

$$=\frac{1-(-1)}{3}$$
$$=\frac{1+1}{3}$$
$$=\frac{2}{3}$$

In the work space provided, write the process used to subtract fractions with like denominators in your own words.

Example 3: Study the following problems and then answer the question that follows.

$$\frac{2}{x}+\frac{3}{x}=\frac{2+3}{x}$$
$$=\frac{5}{x}$$

$$\frac{2}{x+3}+\frac{3}{x+3}=\frac{2+3}{x+3}$$
$$=\frac{5}{x+3}$$

$$\frac{a}{5}+\frac{7}{5}=\frac{a+7}{5}$$

$$\frac{x^2}{x+1}+\frac{1}{x+1}=\frac{x^2+1}{x+1}$$

$$\frac{2x}{x+3}+\frac{6}{x+3}=\frac{2x+6}{x+3}$$
$$=\frac{2(x+3)}{x+3}$$
$$=\frac{2\cancel{(x+3)}}{\cancel{(x+3)}}$$
$$=2$$

In the work space provided, write the process used to subtract rational expressions with like denominators in your own words.

The process used to add and subtract rational expressions with the same denominator follows the same process as adding or subtracting numerical fractions.

Example 4: Study the following problems and then answer the question that follows.

$$\frac{2}{x}-\frac{3}{x}=\frac{2-3}{x}$$
$$=\frac{-1}{x}$$

$$\frac{5}{x+3}-\frac{x}{x+3}=\frac{5-x}{x+3}$$

$$\frac{a}{5}-\frac{7}{5}=\frac{a-7}{5}$$

$$\frac{x^2}{x+1}-\frac{1}{x+1}=\frac{x^2-1}{x+1}$$

$=\dfrac{(x+1)(x-1)}{x+1}$ *Note : We factored x^2-1 into $(x+1)(x-1)$ because it is a "difference of squares".

$=\dfrac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$ * When we factor everything from either the numerator or the denominator it does not become 0, but it is 1.

That is because factoring is division. If you divide 5 by 5 the answer is 1.
Well, if we divide the $(x+1)$ by $(x+1)$ we get 1 as our new denominator.

$$=x-1$$

$$\frac{2x}{x-3}-\frac{6}{x-3}=\frac{2x-6}{x-3}$$
$$=\frac{2(x-3)}{x-3}$$
$$=\frac{2\cancel{(x-3)}}{\cancel{(x-3)}}$$
$$=2$$

$$\frac{n^2}{n-3}-\frac{6n-9}{n-3}=\frac{n^2-(6n-9)}{n-3}$$

$$=\frac{n^2-6n+9}{n-3}$$ * We factored the numerator into (x - 3)(x - 3)

$$=\frac{(n-3)(n-3)}{n-3}$$

$$=\frac{\cancel{(n-3)}(n-3)}{\cancel{(n-3)}}$$

$$=n-3$$

In the work space provided, write the process used to add rational expressions with like denominators in your own words.

To add or subtract rational expressions with the same denominator:

- Add or subtract the algebraic expressions in the numerators. Be careful when simplifying an algebraic expression that follows a subtraction sign.
- Keep the same denominator and place the simplified numerator over it.
- Simplify the rational expression to lowest terms by factoring out common factors.

Add the following rational expressions showing your work.

1. $\frac{1}{x}+\frac{3}{x}$

2. $\frac{5}{c+4}+\frac{3}{c+4}$

3. $\frac{-x}{x+1}+\frac{x}{x+1}$

4. $\frac{2}{a}+\frac{b}{a}$

5. $\frac{x^2}{x+2}+\frac{4x+4}{x+2}$

6. $\frac{b}{b^2-9}+\frac{-3}{b^2-9}$

Subtract the following rational expressions showing your work.

1. $\frac{5}{x} - \frac{6}{x}$

2. $\frac{5}{c+4} - \frac{3}{c+4}$

3. $\frac{x}{x+1} - \frac{1}{x+1}$

4. $\frac{x}{x^2-16}-\frac{4}{x^2-16}$

5. $\frac{x^2}{x-2}-\frac{2x+8}{x-2}$

6. $\frac{b}{b^2-36}-\frac{6}{b^2-36}$

Simplifying Algebraic Expressions
Assessment 2

Add the following rational expressions showing your work.

1. $\dfrac{2}{a}+\dfrac{4}{a}$

2. $\dfrac{6}{n+4}+\dfrac{2}{n+4}$

3. $\dfrac{-2x}{x+1}+\dfrac{3x}{x+1}$

4. $\frac{x^2}{x+3}+\frac{6x+9}{x+3}$

5. $\frac{b}{b^2-25}+\frac{-5}{b^2-25}$

Subtract the following rational expressions showing your work.

6. $\frac{4}{x}-\frac{10}{x}$

7. $\dfrac{17}{r+5} - \dfrac{12}{r+5}$

8. $\dfrac{x}{x+1} - \dfrac{-1}{x+1}$

9. $\dfrac{x}{x^2-36} - \dfrac{6}{x^2-36}$

10. $\frac{x^2}{x-3} - \frac{-2x+15}{x-3}$

In the space below, write the process for simplifying rational expressions with like denominators, in your own words.

Extensions

These websites will provide additional explanation and interactive examples with multiplying and dividing polynomials.

- Multiplying Algebraic Expressions
  http://www.wisc-online.com/objects/index_tj.asp?objID=GEM1904
- Dividing Algebraic Expressions
  http://www.wisc-online.com/objects/index_tj.asp?objID=GEM2104

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards
2008 The Final Report of the National Mathematics Advisory Panel
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools

Solving Systems of Equations
An ADE Mathematics Lesson
Days 28-35

Author	ADE Content Specialists
Grade Level	9th grade
Duration	Eight days

Aligns To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 2: Functions and Relationships
PO 4. Use equations, graphs, tables, descriptions, or sets of ordered pairs to express a relationship between two variables.
PO 5. Recognize and solve problems that can be modeled using a system of two equations in two variables.
Concept 3: Algebraic Representations
PO 1. Create and explain the need for equivalent forms of an equation or expression.
PO 7. Solve systems of two linear equations in two variables.
Concept 4: Analysis of Change
PO 1. Determine the slope and intercepts of the graph of a linear function, interpreting slope as a constant rate of change.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO 5. Graph a linear equation or linear inequality in two variables.
PO 7. Determine the solution to a system of linear equations in two variables from the graphs of the equations.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Connects To

Mathematics HS:
Strand 3: Patterns, Algebra, and Functions
Concept 2: Functions and Relationships
PO 1. Sketch and interpret a graph that models a given context, make connections between the graph and the context, and solve maximum and minimum problems using the graph.
PO 7. Determine domain and range of a function from an equation, graph, table, description, or set of ordered pairs.
Concept 3: Algebraic Representations
PO 3. Write an equation given a table of values, two points on the line, the slope and a point on the line, or the graph of the line.
PO 4. Determine from two linear equations whether the lines are parallel, perpendicular, coincident, or intersecting but not perpendicular.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO 6. Describe how changing the parameters of a linear function affect the shape and position of its graph.

Strand 5: Structure and Logic
Concept 1: Algorithms and Algorithmic Thinking
PO 1. Select an algorithm that explains a particular mathematical process; determine the purpose of a simple mathematical algorithm.

Overview
It is often necessary to solve two equations with two unknowns. Many problems require the use of two variables. In this lesson, you will learn how to solve systems of equations by graphing, substitution, and elimination. You will learn how to set up and solve contextual problems that involve two equations with two unknowns.

Purpose
Solving systems of equations allows you to solve problems that involve more than one unknown. This lesson shows that there are many different ways to solve systems of equations.

Materials
- Systems of equations worksheets
- Ruler
- Graph paper

Objectives
Students will:
- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by elimination/linear combination.
- Set up and solve contextual problems that involve two variables.

Lesson Components

Prerequisite Skills: To complete this lesson, you need to be proficient at graphing and solving equations with one or two unknowns. This lesson builds on skills of setting up equations from contextual problems. During the lesson you will identify parallel lines, intersecting lines, points of intersection, and lines that are coincident (lie on the same line). Prior experience with finding the slope of a line is needed.

Vocabulary: *equation, system of equations, simultaneous equations, graphing, no solution, one solution, infinite number of solutions, substitution, elimination, system of inequalities*

Session 1: Solving Systems of Equations using Graphing (1 day)

1. Use graphing to solve systems of two linear equations in two variables.

Session 2: Solving Systems of Equations using Substitution (2 ays)

1. Use substitution to solve systems of two linear equations in two variables.

Session 3: Solving Systems of Equations using Elimination (2 days)

1. Use elimination/linear combination to solve systems of two linear equations in two variables.

Session 4: Using Systems of Equations to Solve Problems (2 days)

1. Set up and solve contextual problems that involve systems of two linear equations in two variables.

Session 5: Solving Systems of Equations (1 day)

1. Choose any method to solve systems of two linear equations in two variables.

Assessment

There are multiple assessments on solving systems of two linear equations in two variables. These assessments will help you identify errors before you move on the next lesson.

Solving Systems of Equations
Overview

We can solve a system of equations in many ways. A **system of equations** is a set of two or more equations that must all be true for the same value(s) (note: also referred to as simultaneous equations). For this lesson, we will graph a system containing only two equations. A **solution to a system of equations** is the value(s) that hold true for all equations in the system.

A system of equations may have **no solution**, **one solution**, or an **infinite number of solutions**. The system solution methods can include but are not limited to graphical, elimination/linear combination, and substitution. Systems can be written algebraically or can be represented in context.

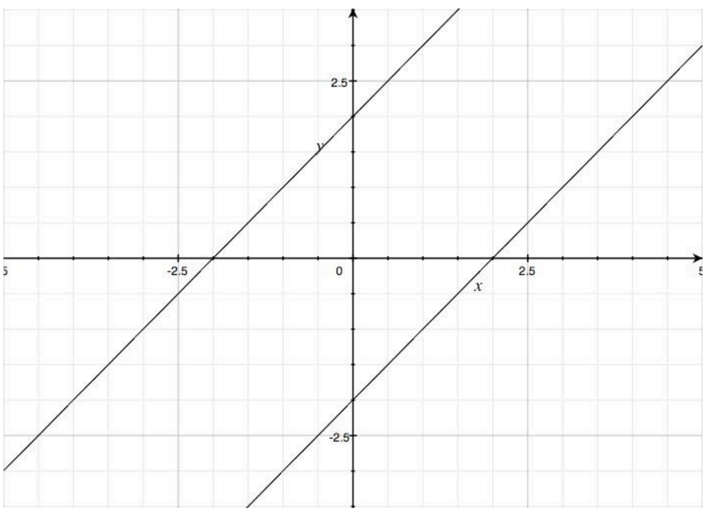
We will learn how to solve a system of two linear equations by graphing in Session 1 and by substitution in Session 2. We will learn how to solve a system of two linear equations by elimination/linear combination in Session 3. We will learn how to solve contextual problems that involve setting up a system of equations in Session 4. In Session 5, you must determine which method to use to solve the system of equations. There is not always a best way to solve a system of equations. Often, it is just personal preference and you must determine which way works best for you to solve the system of equations.

Solving Systems of Equations
Session 1 – Solving Systems of Equations using Graphing

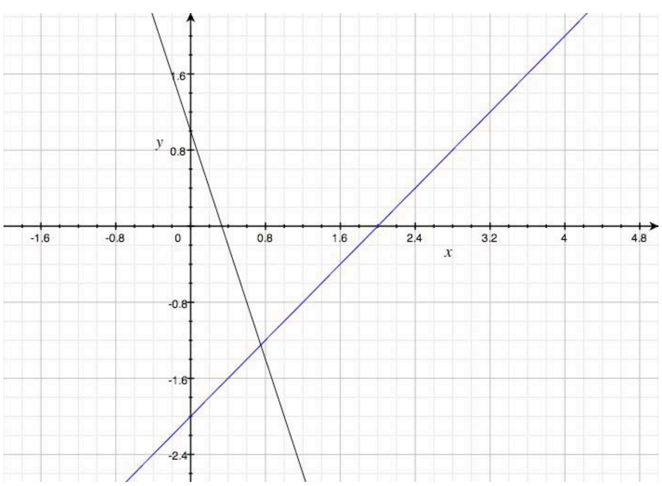
For this session, we solve a system of two linear equations by graphing each line.

When we look at the lines that are graphed, we can determine if they do not meet at all, if they meet in one place, or if they are the same line and therefore meet in all places.

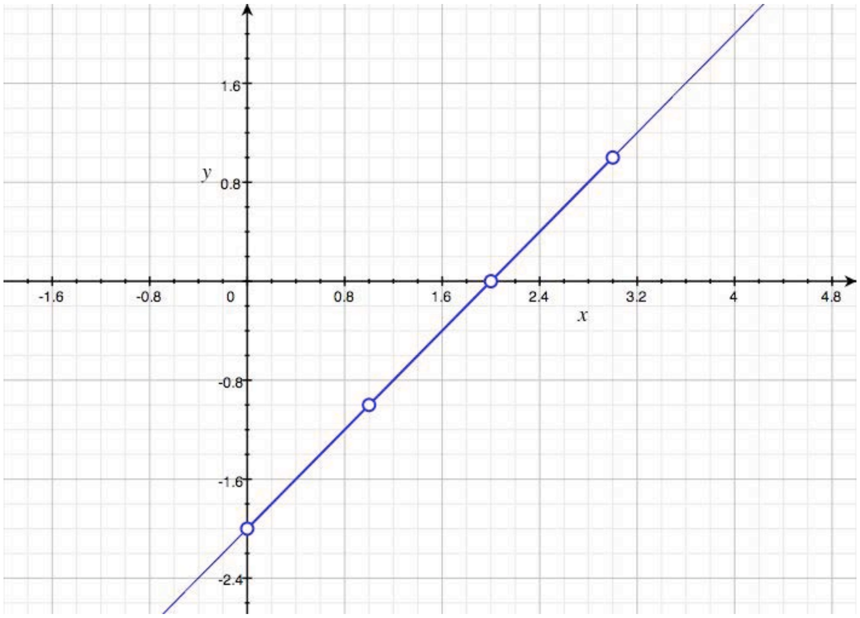
Study the graph of two parallel lines below. How many times do the lines intersect?



Study the graph of two intersecting lines below. How many times do the lines intersect?



Study the graph of two coincident lines (lines that share the same location) below. How many times do the lines intersect?



The definition of a solution for a system of linear equations is, the value(s) that hold true for all equations in the system, the value(s) the lines have in common. Let's investigate what a solution would look like on a graph.

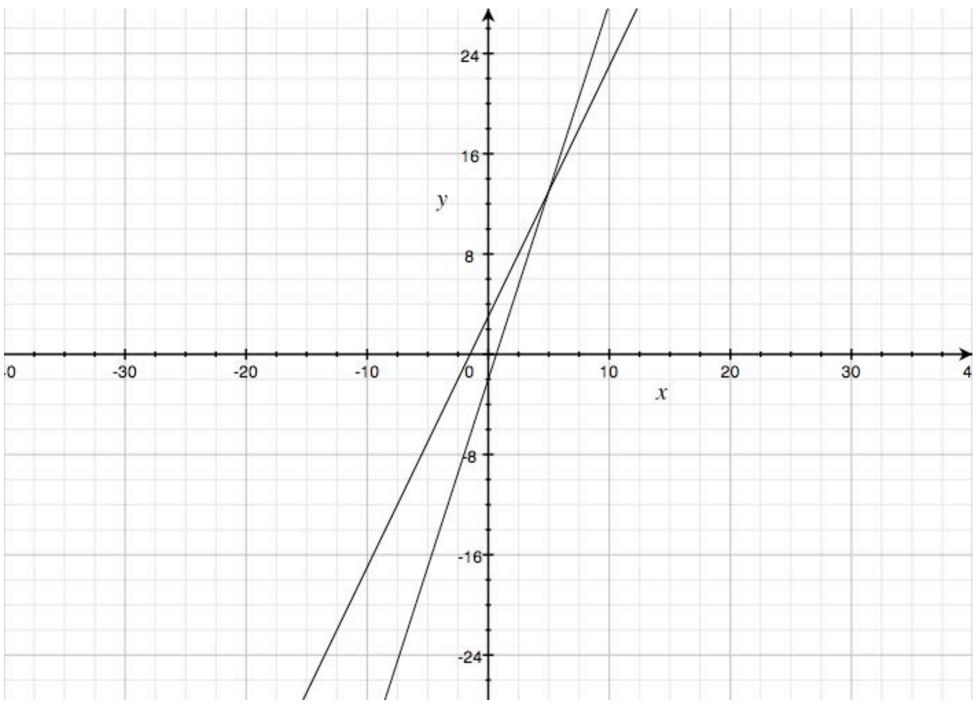
Fill in the tables below.

y = 3x -2	
x	y
0	-2
1	1
2	
3	
4	
5	
6	

y = 2x + 3	
x	y
0	3
1	5
2	
3	
4	
5	
6	

What value do both equations have in common?

Now let's look at the graph of the two lines: y=3x-2 and y=2x+3.



Locate that value the equations had in common. What do you notice?

Now let's think back to the graphs we started with.

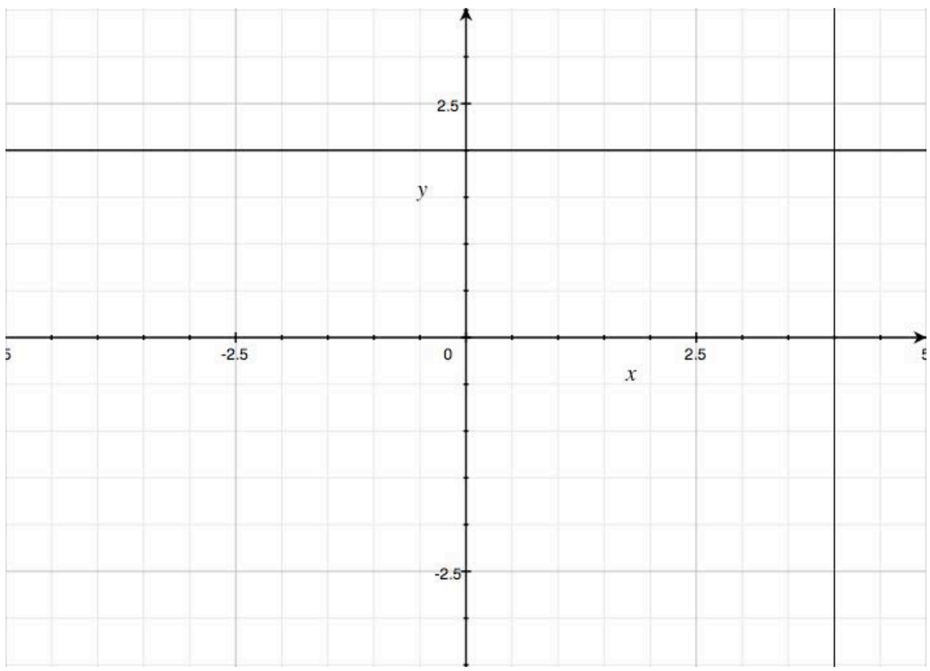
Use what you learned to fill in the table.

Number of solutions	Lines are	Slopes of the lines
	Parallel	The same
	Intersecting	Different
	Coincident	The same

Consider the following examples:

Example 1: Solve the following system of equations by graphing: $\begin{cases} y = 2 \\ x = 4 \end{cases}$

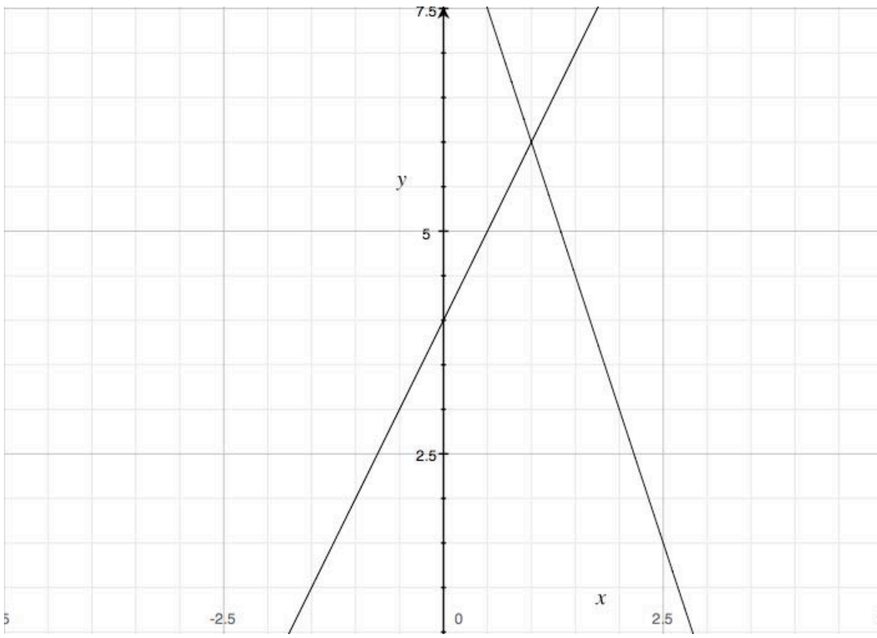
- Graph each equation on the same set of axes.
- After graphing the equations, look to see if the lines intersect.
- If they intersect, note the location where they intersect.
- The place where the lines intersect is the solution to the system of equations.



Study the graphs of $y = 2$ and $x = 4$. The graph of $y = 2$ is a horizontal line with a slope of 0. The graph of $x = 4$ is a vertical line with an undefined slope. The lines intersect at (4, 2). The point (4, 2) is the solution to the system of equations $\begin{cases} y = 2 \\ x = 4 \end{cases}$.

Example 2: Solve the following system of equations by graphing: $\begin{cases} y = 2x + 4 \\ 3x + y = 9 \end{cases}$

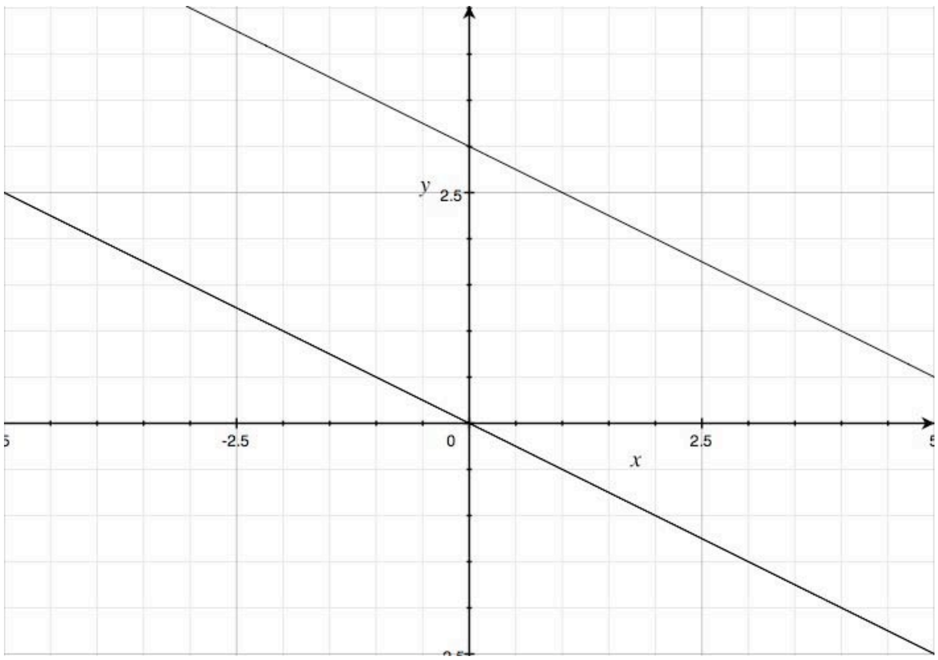
Recall the steps outlined above. Let's graph each equation on the same set of axes. We can graph these equations using a table of values or by placing the equation in slope-intercept form and then graphing.



We observe (1, 6) is the Point of Intersection, the solution for the system of equations.

Example 3: Solve the following system of equations by graphing: $\begin{cases} y = -\dfrac{1}{2}x \\ x + 2y = 6 \end{cases}$

We can graph these equations using a table of values or by placing the equation in slope-intercept form and then graphing.



Study the graphs of these equations. It appears that the lines are parallel. If we place the second equation in slope-intercept form, we can prove what we observe graphically.

$$x + 2y = 6$$

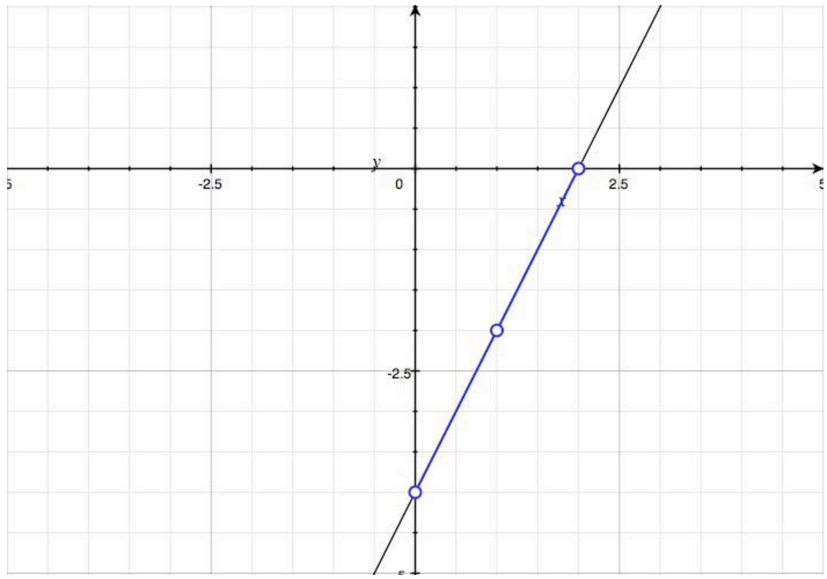
$$2y = -x + 6 \qquad \text{Subtract x from both sides.}$$

$$\frac{2y}{2} = -\frac{x}{2} + \frac{6}{2} \qquad \text{Divide all terms by 2}$$

$$y = -\frac{1}{2}x + 3 \qquad \text{Simplify}$$

Both equations have the same slope proving that <u>the lines are parallel</u>. Therefore there is no point of intersection and <u>there is no solution</u> to this system of equations.

Example 4: Solve the following system of equations by graphing: $\begin{cases} 2x - y = 4 \\ 4x - 8 = 2y \end{cases}$



When we graph these equations, we observe that we have the same line. We can manipulate the second equation to look like the first equation in order to prove they are coincident.

$$4x - 8 = 2y$$

$$\frac{4x-8}{2} = \frac{2y}{2} \quad \text{Divide all terms by 2}$$

$$2x - 4 = y \quad \text{Simplify}$$

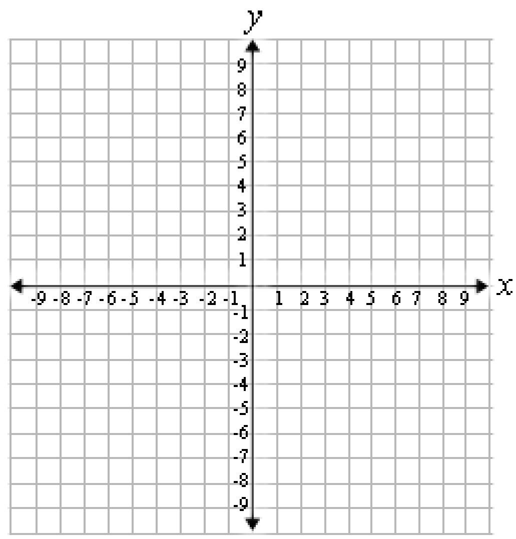
$$2x = y + 4 \quad \text{Add four to both sides}$$

$$2x - y = 4 \quad \text{Subtract y from both sides}$$

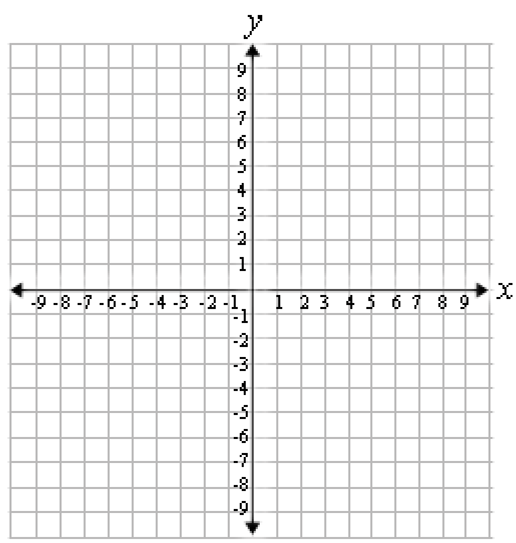
Therefore, any ordered pair that will work in the first equation, $2x - y = 4$, will also work in the second equation, $4x - 8 = 2y$. We have <u>an infinite number of solutions</u>.

Solve these systems of equations by <u>graphing</u>. Indicate the solution to the systems of equations and justify your answer.

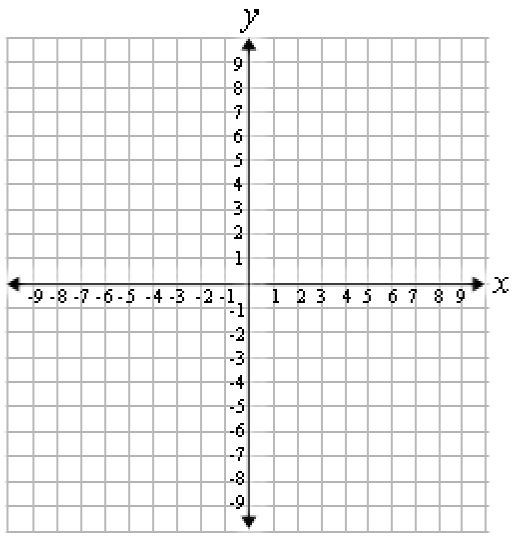
1. Solve the following system of equations: $\begin{cases} x = -3 \\ y = 1 \end{cases}$



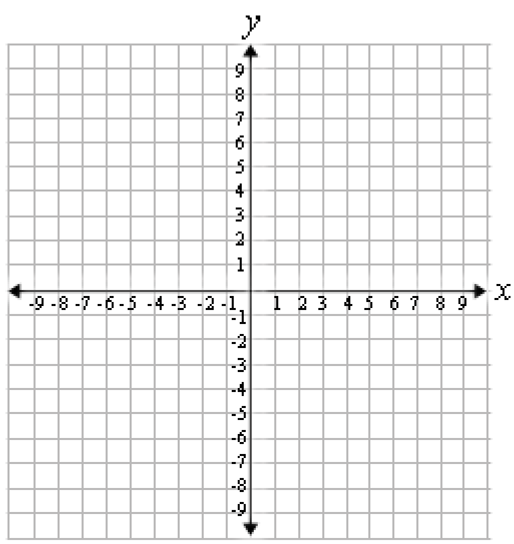
2. Solve the following system of equations: $\begin{cases} x + y = 5 \\ 2y - 4 = x \end{cases}$



3. Solve the following system of equations: $\begin{cases} x + 2y = 4 \\ 2y = 6 - x \end{cases}$



4. Solve the following system of equations: $\begin{cases} 3x - y = 4 \\ 6x + 2y = -8 \end{cases}$



Solving Systems of Equations
Session 2 – Solving Systems of Equations using Substitution

We can solve a system of equations using **substitution**. To solve a system of two linear equations by substitution, follow these steps:

- Solve one equation for one unknown.
- Substitute the value for the unknown in the other equation.
- Simplify the equation and solve for the unknown variable.
- Substitute this value into one of the equations and solve for the other variable.
- Write your solution as an ordered pair.
- Check your answer in both original equations. It is important that you check your answer in both original equations as you may have made a mistake when simplifying your work. If you substitute in at this point, and do the algebra correctly, you may think you have come up with a correct answer when you have not. It is also important that you check your answer in both original equations as your answer may work in one equation but not the other.

Example 1: Solve the following system of equations using **substitution:** $\begin{cases} y = x + 2 \\ 4x + y = 7 \end{cases}$

Step 1: Solve one equation for one unknown. In this system, we have one equation already solve for one variable, $y = x + 2$.

Step 2: Substitute the value for y in the second equation.

$$4x + (x + 2) = 7$$

Step 3: Simplify the resulting equation.

$4x + (x + 2) = 7$	
$(4x + x) + 2 = 7$	Associative Property
$5x + 2 = 7$	Simplify
$5x = 5$	Subtract 2 from both sides
$x = 1$	Divide both sides by 5

Step 4: Now substitute this in one equation and solve for *y* (you can choose either equation. We chose y = x + 2).

$y = x + 2$	
$y = 1 + 2$	Substitute $x = 1$
$y = 3$	Simplify

Step 5: Write your answer as an ordered pair.

The answer to the system of equations is (1, 3) because x = 1 and y = 3 in both equations.

Step 6: Check your answer in both original equations.

$$\begin{aligned} y &= x + 2 \\ 3 &= 1 + 2 \\ 3 &= 3 \end{aligned}$$

$$\begin{aligned} 4x + y &= 7 \\ 4(1) + 3 &= 7 \\ 4 + 3 &= 7 \\ 7 &= 7 \end{aligned}$$

Example 2: Solve the following system of equations using **substitution:** $\begin{cases} 2x - y = 6 \\ 3x + y = 4 \end{cases}$

Step 1: Solve one equation for one unknown. You can choose either equation.
We chose 3x + y = 4.

$$\begin{aligned} &3x + y = 4 \\ &y = 4 - 3x \qquad \text{Subtract 3 x from both sides} \end{aligned}$$

Step 2: Substitute the value for y in the second equation.

$$\begin{aligned} &2x - y = 6 \\ &2x - (4 - 3x) = 6 \end{aligned}$$

Step 3: Simplify the resulting equation.

$$\begin{aligned} &2x - (4 - 3x) = 6 \\ &2x - 4 + 3x = 6 && \text{Distribute the negative through the parentheses} \\ &5x - 4 = 6 && \text{Combine like terms} \\ &5x = 10 && \text{Add 4 to both sides} \\ &x = 2 && \text{Divide both sides by 5} \end{aligned}$$

Step 4: Now substitute this value into one equation and solve for *y.* You can choose either equation. We chose 2x – y = 6.

$$\begin{aligned} &2x - y = 6 \\ &2(2) - y = 6 \\ &4 - y = 6 \\ &-y = 2 \\ &y = -2 \end{aligned}$$

Step 5: Write your answer as an ordered pair. The answer to the system of equations is $(2,-2)$.

Step 6: Check your answer in both original equations.

$$\begin{aligned} &2x - y = 6 \\ &2(2) - (-2) = 6 \\ &4 + 2 = 6 \\ &6 = 6 \end{aligned} \qquad \begin{aligned} &3x + y = 4 \\ &3(2) + (-2) = 4 \\ &6 - 2 = 4 \\ &4 = 4 \end{aligned}$$

Example 3: Solve the following system of equations using **substitution:** $\begin{cases} y = x + 3 \\ y - x = 5 \end{cases}$

Step 1: Solve one equation for one unknown. Since the first equation, $y = x + 3$,is already solved for y, let's use this equation to substitute into the second equation, $y - x = 5$.

Step 2: Substitute the value for y in the second equation.

$$\begin{aligned} &y - x = 5 \\ &(x + 3) - x = 5 \end{aligned}$$

Step 3: Simplify the resulting equation.

$$\begin{aligned} &(x+3) - x = 5 \\ &x - x + 3 = 5 \\ &3 = 5 \end{aligned}$$

Since we know that 3 does not equal 5, we know that there is no solution to this equation.

Step 4: The lines must be parallel in this system of equations since there is no solution. If we put the second equation in slope-intercept form, we can determine if the lines are parallel.

$$\begin{aligned} &y - x = 5 \\ &y = x + 5 \qquad \text{the second equation in slope - intercept form} \end{aligned}$$

$$\begin{aligned} &\mathrm{y = x + 3} \qquad \text{first original equation} \\ &\mathrm{y = x + 5} \qquad \text{second original equation in slope - intercept form} \end{aligned}$$

Step 5: The coefficient of x in both equations is 1. The slope is 1 for both equations. Lines with the same slope and different y-intercepts are parallel. Therefore, there is no solution because the lines will never intersect.

The answer to the system of linear equations is, "**no solution**".

Step 6: You can check the solution by graphing each line and observing that they are parallel or by finding the slope as we did in step 4.

Example 4: Solve the following system of equations using **substitution:** $\begin{cases} 3x - 5y = 15 \\ x - 5 = y \end{cases}$

Step 1: Solve one equation for one unknown. Let's solve the second equation for x and use it to substitute into the first equation.

$$x - 5 = y$$
$$x = y + 5 \qquad \text{add 5 to both sides}$$

Step 2: Substitute the value for x into the first equation.

$$3x - 5y = 15$$
$$3(y+5) - 5y = 15$$

Step 3: Simplify the resulting equation.

$$3(y+5) - 5y = 15$$
$$3y + 15 - 5y = 15$$
$$-2y = 0$$
$$y = 0$$

Step 4: Now substitute this in one equation and solve for y. (you can choose either equation to substitute the value into. We chose the second equation.)

$$x - 5 = y$$
$$x - 5 = 0$$
$$x = 5$$

Step 5: The answer to the system of equations is (5, 0).

Step 6: Check your answer in both original equations.

$$3x - 5y = 15$$
$$3(5) - 5(0) = 15$$
$$15 - 0 = 15$$
$$15 = 15$$

$$x - 5 = y$$
$$5 - 5 = 0$$
$$0 = 0$$

Example 5:

Solve the following system of linear equations using **substitution:** $\begin{cases} 2x+5=2y-1 \\ 3x-2y=7 \end{cases}$

Step 1: Solve one equation for one unknown. Let's solve the first equation for x and use it to substitute into the second equation.

$$\begin{aligned} &2x+5=2y-1 \\ &2x=2y-6 \qquad \text{subtract 5 from both sides} \\ &x=y-3 \qquad \text{divide all terms by 2} \end{aligned}$$

Step 2: Substitute the value for x into the second equation.

$$3(y-3)-2y=7$$

Step 3: Simplify the resulting equation.

$$\begin{aligned} &3(y-3)-2y=7 \\ &3y-9-2y=7 \\ &3y-2y-9=7 \\ &y-9=7 \\ &y=16 \end{aligned}$$

Step 4: Now substitute the value for y into one equation and solve for x. (You can choose either equation to substitute the value into. We chose the second equation.)

$$\begin{aligned} &3x-27=7 \\ &3x-2(16)=7 \\ &3x-32=7 \\ &3x=39 \\ &x=13 \end{aligned}$$

Step 5: The answer to the system of equations is (13, 16).

Step 6: Check your answer in both original equations.

$$\begin{aligned} &2x+5=2y-1 \\ &2(13)+5=2(16)-1 \\ &26+5=32-1 \\ &31=31 \end{aligned}$$

$$\begin{aligned} &3x-2y=7 \\ &3(13)-2(16)=7 \\ &39-32=7 \\ &7=7 \end{aligned}$$

Solve these systems of two linear equations using <u>substitution</u>. Indicate the solution to the systems of equations and check your answer if possible.

1. Solve the following system of linear equations : $\begin{cases} x = -1 \\ y = 4 \end{cases}$

2. Solve the following system of linear equations : $\begin{cases} x - 2y = 5 \\ 2x + y = 15 \end{cases}$

3. Solve the following system of linear equations : $\begin{cases} y - x = -1 \\ x + y = -5 \end{cases}$

4. Solve the following system of linear equations : $\begin{cases} x + 2y = 4 \\ 2x + 4y = -1 \end{cases}$

5. Solve the following system of linear equations : $\begin{cases} 3x - 3y = -6 \\ x - y = -2 \end{cases}$

Solving Systems of Equations
Session 3 – Solving Systems of Equations using Elimination

We can solve a system of equations using **elimination.** This method is also called **linear combination**. To solve a system of two linear equations using elimination we have to eliminate a variable. We can do that by adding the equations when the coefficient of one variable in one equation is the opposite of the coefficient of the same variable in the second equation.

- If the coefficient of one variable in one equation is the opposite of the same variable in the second equation, add the equations.
- The result will be one equation with one variable.
- Solve the resulting equation for the variable.
- Substitute this value in either original equation and solve for the other variable.
- Check your answer by substituting the solution into both original equations.

Example 1: Solve the following system of equations using elimination. $\begin{cases} x+y=4 \\ x-y=2 \end{cases}$

Step 1: Make sure that the coefficient of one variable in one equation is the opposite of the coefficient of the same variable in the other equation.

The coefficients of y are opposites.

Step 2: Eliminate one variable by adding the equations. The coefficient of the y in the first equation is 1. The coefficient of the y in the second equation is -1. These are opposites so we can add the equations together.

$$\begin{array}{r} x+y=4 \\ \underline{x-y=2} \\ 2x=6 \end{array}$$

Step 3: Solve the resulting equation for the variable.

$$\frac{2x}{2}=\frac{6}{2}$$
$$x=3$$

Step 4: Substitute the value into either original equation and solve for the other variable.

$$x+y=4$$
$$3+y=4$$
$$y=1$$

Step 5: Write the solution as an ordered pair.

The solution is (3, 1)

Step 6: Check your answer by substituting the values into both original equations.

$$\begin{array}{ll} x+y=4 & x-y=2 \\ 3+1=4 & 3-1=2 \\ 4=4 & 2=2 \end{array}$$

Example 2: Solve the following system of equations using elimination. $\begin{cases} 3x+36=5y \\ -9-3x=4y \end{cases}$

Step 1: Make sure that the coefficient of one variable in one equation is the opposite of the coefficient of the same variable in the other equation. If we rewrite the second equation using the commutative property we can better see if we have opposites.

$$3x+36=5y \qquad and \quad -9-3x=4y$$
$$-3x-9=4y$$

Now let's write them on top of each other. We have opposite coefficients for x.

$$3x+36=5y$$
$$-3x-9=4y$$

Step 2: Eliminate one variable by adding the equations. The coefficient of the y in the first equation is 1. The coefficient of the y in the second equation is -1. These are opposites so we can add the equations together.

$$\begin{array}{r} 3x+36=5y \\ \underline{-3x-9=4y} \\ 27=9y \end{array}$$

Step 3: Solve the resulting equation for the variable.

$$27=9y$$
$$3=y$$

Step 4: Substitute the value into either original equation and solve for the other variable.

$$3x+36=5y$$
$$3x+36=5(3)$$
$$3x+36=15$$
$$3x=15-36$$
$$3x=-21$$
$$x=-7$$

Step 5: Write the solution as an ordered pair. The solution is (-7, 3)

Step 6: Check your answer by substituting the values into both original equations.

$$3x+36=5y \qquad -9-3x=4y$$
$$3(-7)+36=5(3) \qquad -9-3(-7)=4(3)$$
$$-21+36=15 \qquad -9+21-12$$
$$15=15 \qquad 12=12$$

Sometime we must use a **linear combination** to make sure that the coefficients of one variable are opposites. To solve a system of two linear equations when the coefficient of a variable in the first equation is not the opposite of the coefficient of the same variable in the second equation, follow these steps which describe a linear combination.

- Choose which variable you would like to eliminate.
- Find the Least Common Multiple (LCM) of that variable by looking at the coefficients of the variable in each equation. For example, if the coefficient of x in one equation is 5 and the coefficient of the x in the second equation is 3, the LCM would be 15.
- Multiply the first equation by the factor that will result in the LCM for the chosen variable.
- Multiply the second equation by the factor that will result in the opposite of the LCM (in this case it would be -15) for the same variable.
- Add the equations which will result in one equation with one variable.
- Solve the equation for the variable.
- Substitute this value into one of the original equations and solve for the other variable.
- Check your solution by substituting the solution into both original equations.

Example 3: Solve the following system of equations using **linear combination**: $\begin{cases} 3x - 5y = 1 \\ x + 2y = 4 \end{cases}$

Step 1: Since the LCM of the coefficients of the variable x is 3, and the LCM of the coefficients of the variable y is 10, we can choose either x or y to eliminate. Let's eliminate x. To make the coefficients of the x term opposites, we need to multiply the second equation by -3. The coefficient of the x in the second equation will become -3. These are opposites so we can add the equations together.

$$\begin{aligned} &x + 2y = 4 \\ &-3(x + 2y) = -3(4) \quad \text{Multiply all terms by - 3} \\ &-3x - 6y = -12 \qquad \text{Simplify} \end{aligned}$$

The rewritten system: $\begin{cases} 3x - 5y = 1 \\ -3x - 6y = -12 \end{cases}$

Step 2: Add the equations and solve for the variable.

$$\begin{cases} 3x - 5y = 1 \\ -3x - 6y = -12 \end{cases}$$
$$\overline{\quad -11y = -11 \quad}$$

Step 3: Solve the resulting equation for the variable.

$$\begin{aligned} -11y &= -11 \\ \frac{-11y}{-11} &= \frac{-11}{-11} \\ y &= 1 \end{aligned}$$

Step 4: Substitute the value for the variable in one of the original equations and solve for the second variable.

$$\begin{aligned} &x + 2y = 4 \\ &x + 2(1) = 4 \\ &x + 2 = 4 \\ &x = 2 \end{aligned}$$

Step 5: The solution to the system of equations is (2, 1).

Step 6: Check the solution in both original equations.

$$\begin{aligned} &3x - 5y = 1 \\ &3(2) - 5(1) = 1 \\ &6 - 5 = 1 \\ &1 = 1 \end{aligned} \qquad \begin{aligned} &x + 2y = 4 \\ &2 + 2(1) = 4 \\ &4 = 4 \end{aligned}$$

Let's solve the same system of equations $\begin{cases} 3x - 5y = 1 \\ x + 2y = 4 \end{cases}$ but eliminate the variable *y* first.

Step 1: Find a linear combination to eliminate y. The LCM of 5 and 2 is 10. We can multiply the first equation by 2 and the second equation by 5 and our *y* coefficients should be -10 and 10.

$$\begin{cases} 3x - 5y = 1 \\ x + 2y = 4 \end{cases}$$

$$2(3x) - 2(5y) = 2(1)$$
$$5(x) + 5(2y) = 5(4)$$

$$\begin{cases} 6x - 10y = 2 \\ 5x + 10y = 20 \end{cases}$$

Step 2: Add the equations and solve for the variable.

$$\begin{cases} 6x - 10y = 2 \\ 5x + 10y = 20 \end{cases}$$
$$\overline{\quad 11x = 22 \quad}$$

Step 3: Solve for x.

$$11x = 22$$
$$\frac{11x}{11} = \frac{22}{11}$$
$$x = 2$$

Step 4: Substitute the value for the variable in one of the original equations and solve for the second variable.

$$3x - 5y = 1$$
$$3(2) - 5y = 1$$
$$6 - 5y = 1$$
$$-5y = -5$$
$$\frac{-5y}{-5} = \frac{-5}{-5}$$
$$y = 1$$

Step 5: The solution to the system of equations is (2, 1).

Step 6: Check the solution in both original equations.

$$3x - 5y = 1$$
$$3(2) - 5(1) = 1$$
$$6 - 5 = 1$$
$$1 = 1$$

$$x + 2y = 4$$
$$2 + 2(1) = 4$$
$$4 = 4$$

Note – We arrived at the very same solution whether we eliminated x first or y first.

Example 4: Solve the following system of equations by **elimination**: $\begin{cases} 6x-3y=0 \\ x+y=1 \end{cases}$

Step 1: Find a linear combination to eliminate y. The LCM of 3 and 1 is 3.

$$\begin{cases} 6x-3y=0 \\ x+y=1 \end{cases}$$

$$\begin{aligned} &6x-3y=0 \\ &3(x)+3(y)=3(1) \end{aligned}$$

$$\begin{cases} 6x-3y=0 \\ 3x+3y=3 \end{cases}$$

Step 2: Add the equations and solve for the variable.

$$\begin{array}{c} \begin{cases} 6x-3y=0 \\ 3x+3y=3 \end{cases} \\ \hline 9x=3 \end{array}$$

Step 3: Solve for x.

$$\begin{aligned} &9x=3 \\ &\frac{9}{9}x=\frac{3}{9} \\ &x=\frac{1}{3} \end{aligned}$$

Step 4: Substitute the value for the variable in one of the original equations and solve for the second variable.

$$\begin{aligned} &6x-3y=0 \\ &6\left(\frac{1}{3}\right)-3y=0 \\ &2-3y=0 \\ &-3y=-2 \\ &\frac{-3y}{-3}=\frac{-2}{-3} \\ &y=\frac{2}{3} \end{aligned}$$

Step 5: The solution to the system of equations is $\left(\frac{1}{3},\frac{2}{3}\right)$.

Step 6: Check the solution in both original equations.

$$6x - 3y = 0$$
$$6\left(\frac{1}{3}\right) - 3\left(\frac{2}{3}\right) = 0$$
$$2 - 2 = 0$$
$$0 = 0$$

$$x + y = 1$$
$$\frac{1}{3} + \frac{2}{3} = 1$$
$$1 = 1$$

Solve this same system of equations $\begin{cases} 6x - 3y = 0 \\ x + y = 1 \end{cases}$ by eliminating "x" first. Remember, you should arrive at the same solution, $\left(\frac{1}{3}, \frac{2}{3}\right)$.

Example 5: Solve the following system of equations by **elimination**: $\begin{cases} 5m + 2n = -8 \\ 4m + 3n = 2 \end{cases}$

1. Explain how to eliminate "m" first in the system of equations above.

2. Explain how to eliminate "n" first in the system of equations above.

3. In the space below, solve the system of equations by eliminating either variable first. Remember to check your solution in both original equations.

Check: The correct solution to the system of equations $\begin{cases} 5m + 2n = -8 \\ 4m + 3n = 2 \end{cases}$ is (-4, 6).

Example 6: Solve the following system of equations using **elimination**: $\begin{cases} 2x - y = 3 \\ 2y - 3x = -2 \end{cases}$

Sometimes it is necessary to rearrange the terms in an equation to be able to add the equations together. The LCM of y is 2.

Step 1:

$$\begin{cases} 2x - y = 3 \\ 2y - 3x = -2 \end{cases}$$

$\begin{cases} 2x - y = 3 \\ -3x + 2y = -2 \end{cases}$ Rewrite the second equation using the commutative property.

$\begin{array}{l} 2(2x) - 2(y) = 2(3) \\ -3x + 2y = -2 \end{array}$ Multiply all terms in the first equation by 2 so the coefficients of y are opposites.

$\begin{cases} 4x - 2y = 6 \\ -3x + 2y = -2 \end{cases}$ Our rewritten system is shown.

Step 2:

$$\begin{array}{c} \begin{cases} 4x - 2y = 6 \\ -3x + 2y = -2 \end{cases} \\ \hline x = 4 \end{array}$$

Step 3: The equation is already solved for x.

$$x = 4$$

Step 4:

$$\begin{aligned} &2x - y = 3 \\ &2(4) - y = 3 \\ &8 - y = 3 \\ &y = 5 \end{aligned}$$

Step 5: The solution to the system of equations is (4, 5).

Step 6:

$$\begin{aligned} &2x - y = 3 \\ &2(4) - 5 = 3 \\ &8 - 5 = 3 \\ &3 = 3 \end{aligned} \qquad \begin{aligned} &2y - 3x = -2 \\ &2(5) - 3(4) = -2 \\ &10 - 12 = -2 \\ &-2 = -2 \end{aligned}$$

Example 7: Solve the following system of equations using **elimination**: $\begin{cases} 3x+6y=12 \\ 6x+4y=-8 \end{cases}$

Sometimes it is necessary to multiply by a negative number in order to get *opposite* coefficients.

Step 1:

$$\begin{cases} 3x+6y=12 \\ 6x+4y=-8 \end{cases}$$

6 is the LCM for the x coefficients but **BOTH** are positive. I will need to multiply the first equation by -2 to make sure they are opposites. Make sure to multiply all terms in the first equation by 2 so the coefficients of y are opposites.

$$-2(3x+6y)=-2(12)$$
$$-6x-12y=-24$$

$\begin{cases} -6x-12y=-24 \\ 6x+4y=-8 \end{cases}$ Our rewritten system is shown.

Step 2:

$$\begin{cases} -6x-12y=-24 \\ \underline{6x+4y=-8} \end{cases}$$
$$-8y=-32$$

Step 3: Solve for y.

$$-8y=-32$$
$$\frac{-8y}{-8}=\frac{-32}{-8}$$
$$y=4$$

Step 4: Substitute it back into an original equation.

$$6x+4y=-8$$
$$6x+4(4)=-8$$
$$6x+16=-8$$
$$6x=-16-8$$
$$6x=-24$$
$$x=-4$$

Step 5: The solution to the system of equations is (-4, 4).

Step 6: Check the solution by substituting it into both original equations.

$$3x+6y=12$$
$$3(-4)+6(4)=12$$
$$-12+24=12$$
$$12=12$$

$$6x+4y=-8$$
$$6(-4)+4(4)=-8$$
$$-24+16=-8$$
$$-8=-8$$

Solve these systems of equations using <u>elimination</u>. Indicate the solution to the systems of equations and check your answer if possible.

1. Solve the following system of equations : $\begin{cases} 6x + 4y = 24 \\ 2x - 4y = 8 \end{cases}$

2. Solve the following system of equations : $\begin{cases} x - 2y = 5 \\ 2x + y = 15 \end{cases}$

3. Solve the following system of equations : $\begin{cases} c + d = -1 \\ -2c - 3d = 0 \end{cases}$

4. Solve the following system of equations : $\begin{cases} x + 2y = 4 \\ 2x + 4y = -1 \end{cases}$

5. Solve the following system of equations : $\begin{cases} 3m - 3n = -6 \\ m - 2n = -2 \end{cases}$

Solving Systems of Equations
Session 4 – Using Systems of Equations to Solve Problems

It is often necessary to set up two equations with two variables to solve a problem. Each of the problems below can be solved using a system of equations. Follow the steps below to solve each problem.

- Read the problem twice to determine the context of the problem and to identify the key components of the problem. There is very often extraneous (unnecessary) information in the problem so be careful to thoroughly understand the problem.
- Identify the unknown values and assign different variables to each.
- Set up a system of equations to identify each given fact.
- Solve the system of equations using an appropriate method.
- Answer all the questions in the original problem.
- Check your answers in the original problem. You may have set up the equation incorrectly. If you check your answers using the equation you made then it might indicate you have a correct answer when you do not.

Example 1: Find two numbers whose sum is 64 and whose difference is 42.

Solution:

The problem is asking us to find two numbers that when added equal 64 and when subtracted equal 42.

- Let x = one number. Let y = the second number.
- Set up a system of equations based on the information given. $\begin{cases} x+y=64 \\ x-y=42 \end{cases}$
- Solve the system of equations using any appropriate method. In this case, since the coefficients of "y" are opposites, we can use elimination very easily.

$$\begin{cases} x+y=64 \\ x-y=42 \end{cases}$$

$$2x=106$$

$$\frac{2x}{2}=\frac{106}{2}$$

$$x=53$$

$$x+y=64$$

$$53+y=64$$

$$y=11$$

The numbers are 53 and 11. To check the solution, go back to the words of the problem.

Find two numbers whose sum is 64 53+11=64

and whose difference is 42 53-11=42

We know that we have the correct answer.

Example 2:

The sum of two numbers is 16. Three times the greater number equals the sum of four times the lesser number increased by 6. Find the numbers.

Solution: The problem is asking us to find two numbers that when added equal 16. We know that multiplying 3 times the larger number is 4 times the smaller number plus 6.

- Let x = larger number.   Let y = the smaller number.
- Set up a system of equations based on the information given. The first equation will be $x + y = 16$. The second equation is $3x = 4y + 6$. Our system of equations is

  $$\begin{cases} x + y = 16 \\ 3x = 4y + 6 \end{cases}$$

- Let's use substitution to solve this problem.

$$x + y = 16$$
$$x = 16 - y$$

$$3x = 4y + 6$$
$$3(16 - y) = 4y + 6$$
$$48 - 3y = 4y + 6$$
$$42 - 3y = 4y$$
$$42 = 7y$$
$$y = 6$$

$$x + y = 16$$
$$x + 6 = 16$$
$$x = 10$$

The numbers are 10 and 6.

Check the answers in the original word problem. Two numbers total 16 $10 + 6 = 16$

Three times the larger number is 4 times the smaller number increased by 6. 3 times 10 is 30. 4 times 6 is 24. 24 increased by 6 is 30.

30=30

Since the arithmetic works for both facts in the original problem, we know that we have the correct answer.

Example 3: Larissa has $1.95 in quarters and dimes. She has a total of 9 coins. How many coins are quarters and how many are dimes?

Solution: The problem is telling us that Larissa has $1.95 but that she only has quarters and dimes. We are to find the number of each coin that Larissa has.

- Let q represent the number of quarters that Larissa has and d represents the number of dimes that Larissa has.
- Set up the equations. $q + d = 9$ represents the fact that Larissa has 9 coins. Since a quarter is worth $.25 and a dime is worth $.10 we can set up the second equation using this information. $.25q + .10d = 1.95$. Then our system of equations becomes

  $$\begin{cases} q + d = 9 \\ .25q + .10d = 1.95 \end{cases}$$

- Now we can solve this system of equations using any method we wish. Let's use elimination. If we multiply the second equation by -10, we have opposite numbers for the coefficients of d.

$$\begin{cases} q + d = 9 \\ .25q + .10d = 1.95 \end{cases}$$

$$\begin{cases} q + d = 9 \\ -10(.25)q - 10(.10d) = (-10)1.95 \end{cases}$$

$$\begin{cases} q + d = 9 \\ -2.5q - d = -19.5 \end{cases}$$

$$\begin{cases} q + d = 9 \\ -2.5q - d = -19.5 \end{cases}$$

$$-1.5q = -10.5$$

$$\frac{-1.5}{-1.5}q = \frac{-10.5}{-1.5}$$

$$q = 7$$

$$q + d = 9$$

$$7 + d = 9$$

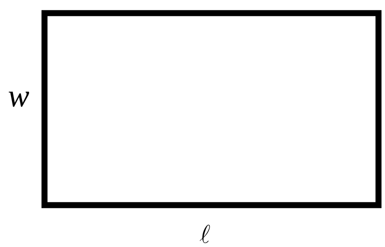
$$d = 2$$

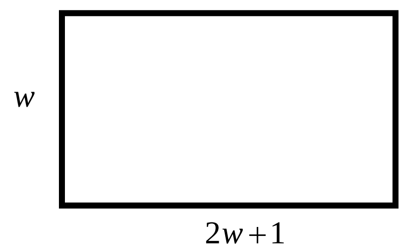
The solution to the problem is that Larissa has 7 quarters and 2 dimes. Check the solution in the original equation. $7 + 2 = 9$. $7(\$.25) = \$1.75$ and $2(\$.10) = \$.20$. $\$1.75 + .20 + \$1.95$. Since the arithmetic fits for the original problem, we know we have the correct solution.

Example 4: The perimeter of a rectangle is 20 inches. The length of the rectangle is one more than twice its width. What is the area of the rectangle?

Solution: This problem is telling us that we have a rectangle and gives us information about the relationship between the width of the rectangle and its length. Note that the problem is asking us to find the area of the rectangle so we must first find the dimensions of it and then find its area. It helps to draw a diagram when solving this type of problem.

- Let w equal the width of the rectangle and l equal the length of the rectangle.
- Now draw a diagram.





- Set up the equations. Perimeter is 2 times the width plus 2 times the length.

$$P = 2w + 2\ell$$

$$20 = 2w + 2\ell$$

$$\ell = 2w + 1$$

- Solve the system of equations. $\begin{cases} 20 = 2w + 2\ell \\ \ell = 2w + 1 \end{cases}$
- Simplify the first equation:

$$20 = 2w + 2\ell$$

$$20 = 2(w + \ell)$$

$$\frac{20}{2} = \frac{2(w + \ell)}{2}$$

$$10 = w + \ell$$

- Now use substitution to solve the system of equations:

$$10 = w + \ell$$

$$10 = w + 2w + 1$$

$$10 = 3w + 1$$

$$9 = 3w$$

$$w = 3$$

- Substitute in this value to find the length.

  $2w + 1 = \ell \qquad 2(3) + 1 = 7 \qquad 6 + 1 = \ell \qquad 7 = \ell$

  The dimensions of the rectangle are 3 by 7. Check this solution first in the original problem before finding the area of the rectangle. Perimeter equals

  $2w + 2\ell = 20 \qquad 2(3) + 2(7) = 20 \qquad 2w + 1 = \ell \qquad 2(3) + 1 = 7$

- Now find the area of the rectangle. $A = \ell \bullet w$ $3 \bullet 7 = 21$ square units.

Solve the following problems using a system of two linear equations. Show your work. Make certain to answer all the questions posed in the problems.

1. The tens digit of a two-digit number is two more than twice the units digit. The sum of the digits is 11. Use a system of equations and any method to find the two-digit number.

2. Billy bought a pair of shorts and a shirt. His bill without tax was $35.00. The shirt cost $10 less than twice the cost of the shorts. What was the cost of each item of clothing?

3. At the supermarket this week, butter cost four times more than margarine. Together, the margarine and butter cost $7.25. What did the butter and the margarine cost individually?

Solving Systems of Equations
Session 5 – Solving Systems of Equations

Solve the following system of two linear equations showing your work. You may use any method you wish. Check your solution for at least two of these systems graphically.

1. $\begin{cases} x+5y=4 \\ 3x+15y=-1 \end{cases}$

2. $\begin{cases} 4a+5b=6 \\ 6a-9b=-2 \end{cases}$

3. $\begin{cases} s=3t-3 \\ 2s-6t-=6 \end{cases}$

4. $\begin{cases} x+y=3 \\ 2y=1-2x \end{cases}$

5. $\begin{cases} \dfrac{1}{2}x+2y=12 \\ x-2y=6 \end{cases}$

6. $\begin{cases} 2x-y=-4 \\ x+y=4 \end{cases}$

7. $\begin{cases} 2x+5y=7 \\ y=-\dfrac{2}{5}x+\dfrac{7}{5} \end{cases}$

8. $\begin{cases} \dfrac{1}{2}x-2y=9 \\ y=3-x \end{cases}$

9. $\begin{cases} 8m - 7n = -10 \\ 4m + 4n = 3 \end{cases}$

10. $\begin{cases} x = 2y \\ 4x + 4y = 3 \end{cases}$

11. $\begin{cases} x + y = \frac{1}{2} \\ x = -2y \end{cases}$

Solving Systems of Equations
Assessment 1

1. Describe how to solve a system of two linear equations by graphing.

2. Describe how to solve a system of two linear equations by substitution.

3. Describe how to solve a system of two linear equations by elimination/linear combination.

4. Explain what the graph shows when there is no solution to a system of two linear equations. Give an example of a system of two linear equations that has no solution (sketch a graph or write equations).

5. Explain what the graph shows when there are an infinite number of solutions to a system of two equations. Give an example of a system of two equations that has an infinite number of solutions (sketch a graph or write equations).

6. Explain what the graph shows when there is one solution to a system of two equations. Give an example of a system of two equations that has exactly one solution (sketch a graph or write equations).

Solving Systems of Equations
Assessment 2

Consider the following system of linear equations: $\begin{cases} 4x + y = 0 \\ x + 2y = -7 \end{cases}$

1. Solve the above system using elimination.

2. Check your answer to #1 using both original equations.

3. Solve the system using substitution.

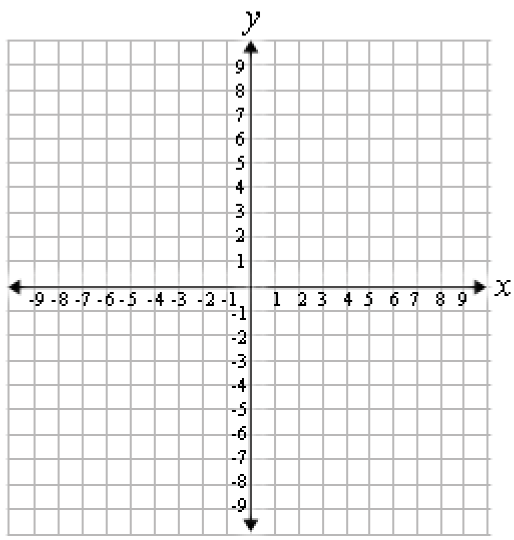
4. Solve the system of equations graphically. Make certain to mark the units on the axes and to write your answer as an ordered pair. Use graph paper to show your work and solution.

Solving Systems of Equations
Assessment 3

Answer the following questions completely **showing your work.**

1. Solve the following system of equations **graphically**.

$$\begin{cases} 3x - y = 4 \\ 6x + 2y = -8 \end{cases}$$



2. Solve the following system of equations using **substitution**.

$$\begin{cases} x - 2y = 5 \\ 3x - 5y = 8 \end{cases}$$

3. Solve the following system of equations using **elimination**.

$$\begin{cases} 5m + 2n = -8 \\ 4m + 3n = 2 \end{cases}$$

4. Find two numbers whose sum is 50 and whose difference is 28. Solve the problem using a system of linear equations.

Extensions

1. Solve systems of two linear inequalities in two variables.

Solving Systems of Equations
Extension

We can also have a system of linear inequalities. We usually solve these systems by graphing each inequality and determining the regions of overlap.

A **solution to a system of linear inequalities** is the set of all values that make each inequality true. When we graph inequalities, remember that a solid line indicates that the line is included in the inequality. A dotted line indicates that the line is not included in the inequality.

To solve a system of linear inequalities, follow these steps

- Graph each line that is part of the system on the same coordinate grid. It is very helpful to graph each inequality using a separate color.
- Graph a solid line if the inequality is $\leq$ or $\geq$ since this indicates that the solution includes the line.
- Graph a dotted line if the inequality is $<$ or $>$ since this indicates that the solution does not include the line.
- Choose one point on the graph. If that point makes the inequality a correct statement, color all points on the side of the line that includes that point.
- If that point makes the inequality an incorrect statement, color all points on the side of the line that does not include that point.
- The region (set of points) that has both shades (where the color overlaps) is the solution set for the system of inequalities.

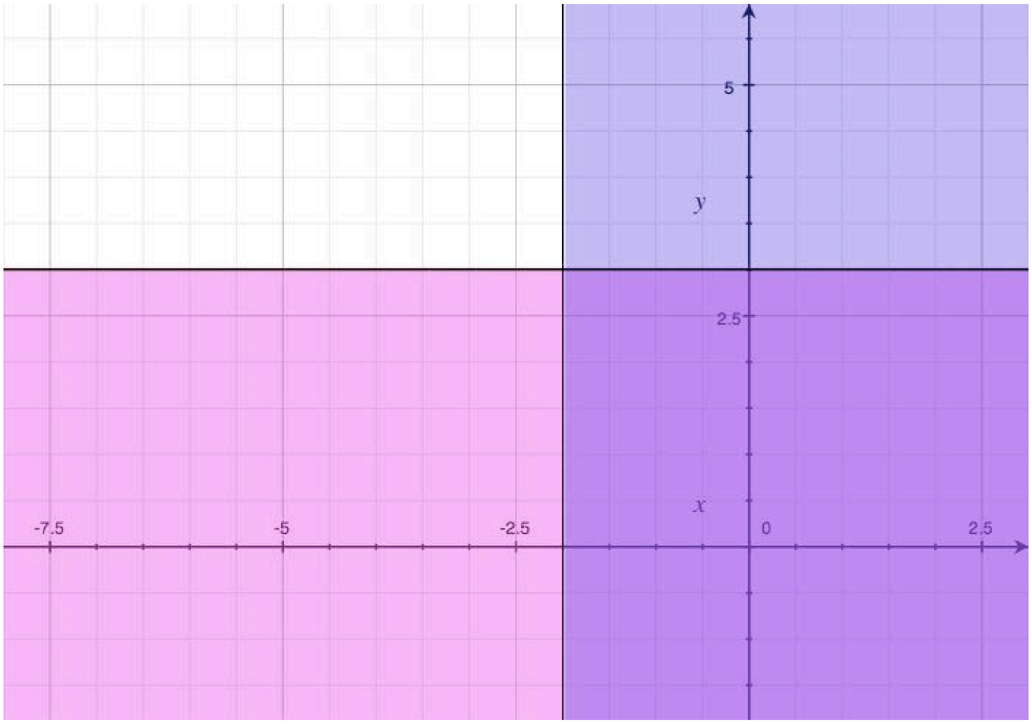
Example 1: Solve the following system of inequalities: $\begin{cases} y \leq 3 \\ x \geq -2 \end{cases}$

Solution:

Since the two inequalities contain ≤ and ≥, we will graph the lines $y = 3$ and $x = -2$ using a solid line. Graph each line on the same coordinate grid indicating which line is which (try using different colors). Choose the point (0, 0) to test each inequality.

Choosing (0, 0) with the inequality $y \leq 3$ means that $0 \leq 3$. This is a true statement so color the side of the line that contains the point (0, 0).

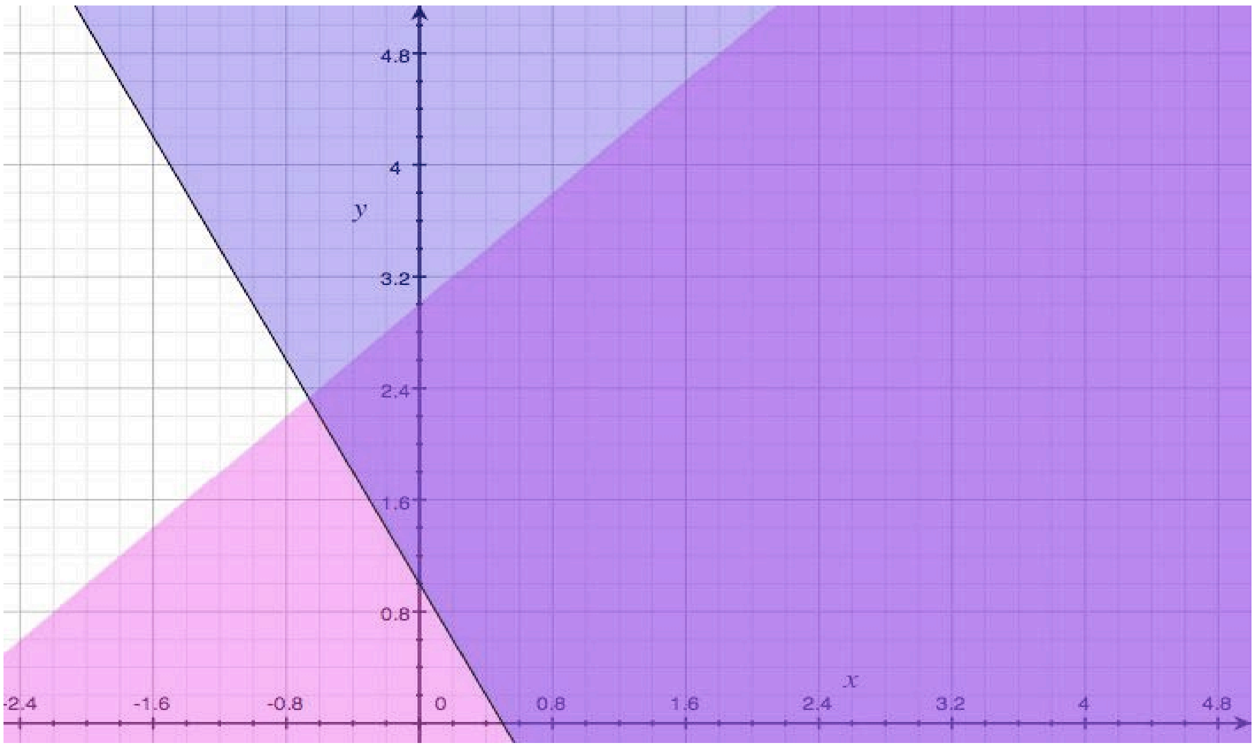
Now try $x \geq -2$ with the same point $0 \geq -2$ This is also true statement so color the side of the line that contains that point.



The solution to the set of inequalities is the set of all points that are both colors.

Example 2: Solve the following system of inequalities: $\begin{cases} y < x+3 \\ y \geq -2x+1 \end{cases}$

Solution: Since the first equation contain <, we will graph the line $y = x+3$ using a dotted line. Since the second equation contains ≥, we will graph the line $y = -2x+1$ using a solid line.



Choose the point (0, 0) as a test point for each inequality.

$y < x+3$

$0 < 0+3$

$0 < 3$

This is true statement so color all the points on the side of the line $y = x+3$ that contains (0, 0).

$y \geq -2x+1$

$0 \geq -2 \bullet 0+1$

$0 \geq 1$

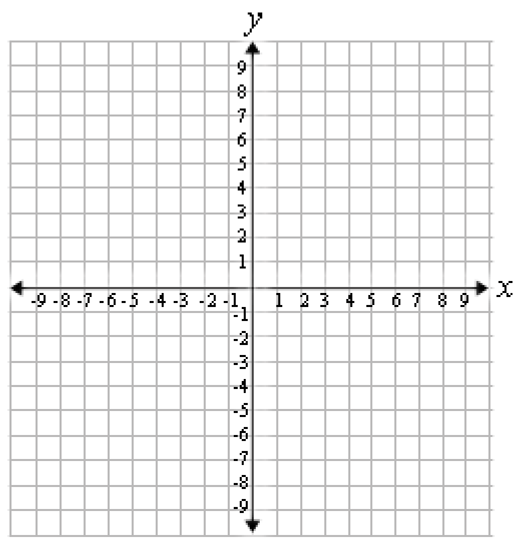
This is not a true statement so color all the points on the side of the line that does not contain the point (0, 0).

The solution to the set of inequalities is the set of all points that are both colors.

Answer the following questions completely <u>showing your work</u>.

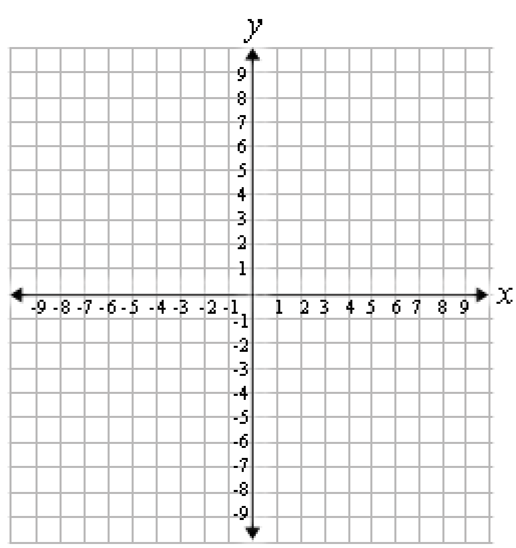
1. Solve the following system of inequalities: $\begin{cases} y \geq -x + 2 \\ y > 2x - 1 \end{cases}$

   Show your work. Use a different color to represent each inequality.



2. Solve the following system of inequalities: $\begin{cases} y < x - 4 \\ y \leq -2x + 3 \end{cases}$

   Show your work. Use a different color to represent each inequality.



Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards
2008 The Final Report of the National Mathematics Advisory Panel
1999 Bringing the NCTM Standards to Life, Exemplary Practices from High Schools

Simplifying Radical Expressions
An ADE Mathematics Lesson
Days 36-40

Author	ADE Content Specialists
Grade Level	9^{th} grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 1: Number Sense
PO 1. Justify with examples the relation between the number system being used (natural numbers, whole numbers, integers, rational numbers and irrational numbers) and the question of whether or not an equation has a solution in that number system.
Concept 2: Numerical Operations
PO 1. Solve word problems involving absolute value, powers, roots, and scientific notation.
PO 3. Calculate powers and roots of rational and irrational numbers.
Concept 3: Estimation
PO 1. Determine rational approximations of irrational numbers.
PO 4. Estimate the location of the rational or irrational numbers on a number line.

Strand 3: Patterns, Algebra, and Functions
Concept 3: Algebraic Representations
PO 8. Simplify and evaluate polynomials, rational expressions, expressions containing absolute value, and radicals.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 2: Number Sense
PO 2. Solve word problems involving absolute value, powers, roots, and scientific notation.
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.
PO 3. Determine when an estimate is more appropriate than an exact answer.

Overview
Many algebraic expressions contain radicals. These radicals may contain numbers and/or variables. Some of these radicals can be simplified prior to simplifying the entire algebraic expression. The value of some radicals needs to be estimated in order to solve problems that are in a contextual format.

Purpose

Working with radical expressions allows you to work with both rational and irrational numbers when they are simplifying algebraic expressions. This lesson provides the foundation for solving algebraic equations that contain radical expressions.

Materials

- Radical expressions worksheets
- Number lines worksheets

Objectives

Students will:

- Simplify numerical radical expressions.
- Estimate the square root of a number to the nearest tenth.
- Solve contextual problems that involve square roots.
- Determine the number system(s) (natural numbers, whole numbers, integers, rational, and irrational numbers) to which the terms in a problem belong.
- Predict whether or not the solution to a problem will be from a different number system(s) than terms in the original problem, justify the prediction, and prove whether or not the prediction was correct.
- Simplify radical expressions that contain both numbers and variables.
- Multiply radical expressions.

Lesson Components

Prerequisite Skills: This lesson builds on skills from earlier grades of simplifying algebraic expressions that did not contain radical expressions. Prior experience with the various number sets including rational, irrational, natural numbers, whole numbers, and integers is helpful. You will need to set up and read Venn Diagrams. You will also need to remember operations with integers.

Vocabulary: *radical, radical expression, radicand, index, square root, cube root, principal square root, natural numbers, whole numbers, integers, rational numbers, irrational numbers.*

Session 1 (1 day)

1. Simplify radicals that contain only numbers in their radicands.
2. Estimates the value of a square root to the nearest tenth.
3. Identify a decimal as representing either a rational or an irrational number and locate the numbers on a number line.

Session 2 (1 day)

1. Simplify radicals that contain variables in the radicand.

Session 3 (2 days)

1. Create a Venn diagram or other graphic organizer that shows the relationship between natural numbers, whole numbers, integers, rational numbers, irrational numbers and real numbers.
2. Identify the number system(s) for terms in a problem, and predict to which number system(s) the solution will belong.
3. Justify their prediction and prove whether or not the prediction was correct.

Session 4 (1 day)

1. Solve problems that contain radicals in contextual situations.

Assessment

There is one assessment for each session to help you identify errors before moving on to the next session.

Radical Expressions
Session 1

A mathematical expression that contains a **radical** (a symbol used to refer to the root of a number or term), $\sqrt{\ }$, is called a **radical expression**. There are rules to simplifying these expressions. Radical expressions include variables and/or numbers under the radical sign. The number, variable, or algebraic expression under the radical sign is called the **radicand**.

Example : In the expression $\sqrt{49}$ **,** 49 is the radicand. In $\sqrt{211}$, 211 is the radicand.

A radical expression can take many forms such as $\sqrt{25}$ or $\sqrt[3]{27}$ or $\sqrt[4]{81}$. In these expressions, 25, 27, and 81 are radicands. The 3 in the second radical expression and the 4 in the third radical expressions are referred to as the **index** of the radical expression. When there is no index written we assume that the index is 2 and we are finding the **square root** of an expression.

$\sqrt{25} = \sqrt[2]{25}$ but we do not write the 2. The index is 2. $\sqrt{25} = 5$ because $5 \bullet 5 = 25$.

If the index is 3, we are trying to find the **cube root** of a number or of an algebraic expression. The index tells us how many repeat factors are needed.

A root is the inverse of an exponent. The inverse of an exponent of 2 is a square root. The inverse of a cube root is raising to the exponent 3. That is important because it let's you know how to simplify a term and remove it from the radical. $\sqrt{9}$ can be rewritten as $\sqrt{3 \bullet 3}$ or $\sqrt{3^2}$.

Since the square root of 3 raised to the second power is being taken, the exponent and the root cancel. That leaves just 3.

$\sqrt[3]{8}$ = 2 because we can factor 8 into $2 \bullet 2 \bullet 2$. Consider $\sqrt[3]{8} = \sqrt[3]{2^3}$. We have three 2's under the radical sign when the index is three so the inverse property allows us to simplify the term to 2. Then $\sqrt[3]{8}$ = 2.

We will only work with square roots in this lesson but it is important that you understand the meaning of the index of a radical expression. You may work with cube roots in the extension of this lesson.

Example 1: Simplify $\sqrt{49}$.

Solution: We can factor 49 to $7 \bullet 7$. Then we can replace the radicand 49 with 7^2 and then simplify. $\sqrt{49}$ = $\sqrt{7^2}$. Therefore, according to the inverse property $\sqrt{49}$ = 7. Every pair of the same numbers or the same variables under the radical sign can be simplified to the number/variable. Note that we can say -7$\bullet$-7 =49 or 7$\bullet$7=49. Therefore, $\sqrt{49}$ = 7 or $\sqrt{49}$ = -7. We can write $\sqrt{49}$ = $\pm$7 as a short way to express both answers. The positive square root of a number is also called the **<u>principal square root</u>**.

The square root of a fraction equals the square root of its numerator divided by the square root of its denominator.

Example 2: Find $\pm\sqrt{\frac{100}{81}}$.

Solution: $\sqrt{100}$ = $\pm$10 $\pm\sqrt{81}$. Therefore $\pm\sqrt{\frac{100}{81}}$ = $\pm\frac{10}{9}$.

(Note that we only write one $\pm$ sign as it would be repetitive to write it more than once.)

Simplify:

1. $\sqrt{225}$

2. $-\sqrt{49}$

3. $\sqrt{121}$

4. $\pm\sqrt{\frac{121}{25}}$

5. $-\sqrt{\frac{49}{169}}$

Example 3: Find $\sqrt{75}$.

Method 1 *(Factor the radicand completely)***:** We can factor 75 as $5 \bullet 5 \bullet 3$.

Then, $\sqrt{75} = \sqrt{5 \bullet 5 \bullet 3}$. Therefore $\sqrt{75} = 5\sqrt{3}$.

Method 2 *(Factor out perfect squares)***:** We can factor 75 as $25 \bullet 3$.

Then, $\sqrt{75} = \sqrt{25 \bullet 3}$. The square root of 25 is 5. Therefore $\sqrt{75} = 5\sqrt{3}$.

Simplify:

1. $\sqrt{200}$

2. $\sqrt{27}$

3. $\sqrt{120}$

Write the missing perfect squares from 1 to 225 in the table below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1						49								225

To estimate the square root of a rational number, determine between which two perfect squares it falls. Then make an estimate based on the proximity of the rational number to these two numbers.

For example, find $\sqrt{200}$ Since 200 falls between 14^2 = 196 and 15^2 = 225, we know that the square root of 200 must lie between 14 and 15. It is probably closer to 14 since 196 is much closer to 200 than to 225. Therefore, a good estimate of $\sqrt{200}$ would be 14.1 or 14.2. <u>Either estimate is acceptable</u>.

Note that the difference between 196 and 225 is 29. We may then consider 29 ÷ 10 = 2.9. Since 2.9 is close to 3, we may increase our estimate by .1 for every 3 units. Therefore, 14.2 may be a slightly closer estimate to the actual square root of 200.

Estimate each square root to the nearest tenth and explain your reasoning.

1. $\sqrt{140}$

2. $\sqrt{62}$

3. $\sqrt{192}$

4. $\sqrt{200}$

5. $\sqrt{50}$

The square root of a number that is a perfect square always represents a rational number. The square root of a number that does not represent a perfect square is always an irrational number.

Examples:

1. $\sqrt{36}$ is a rational number because 36 is a perfect square. $\sqrt{36}$ can be rewritten as 6.
2. $\sqrt{5}$ is an irrational number because 5 is not a perfect square.

Radical Expressions
Assessment 1

Simplify the following radical expressions showing your work.

1. $\sqrt{169}$

2. $-\sqrt{81}$

3. $\pm\sqrt{\dfrac{144}{49}}$

4. $\sqrt{300}$

Complete the following table with the missing perfect squares.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1				25							144			

Which set contains an irrational number?

a. $\{\sqrt{4n^2}$, 0.23, $\dfrac{15}{1}\}$

b. $\{21, 0.3, \dfrac{\sqrt{121}}{2\sqrt{1}}\}$

c. $\{\dfrac{5}{8}, 8, \sqrt{26}\}$

d. $\{0.222\ldots, \sqrt{9}, 15\}$

Radical Expressions
Session 2

We can simplify radical expressions that contain variables by following the same process as we did for radical expressions that contain only numbers. Factor the expression completely (or find perfect squares). For every pair of a number or variable under the radical, they become one when simplified. If a pair does not exist, the number or variable must remain in the radicand.

Example 1: Simplify $\sqrt{x^2y^4}$.

We can represent this in an equivalent form by $\sqrt{x \bullet x \bullet y \bullet y \bullet y \bullet y}$.

We take out one pair of x's and two pairs of y's from the radicand .

$\sqrt{(x \bullet x) \bullet (y \bullet y) \bullet (y \bullet y)}$

$= x \bullet y \bullet y$

$= xy^2$

Example 2: Simplify $\sqrt{36a^3b^4c^2}$

We can represent this by $\sqrt{6 \bullet 6 \bullet a \bullet a \bullet a \bullet b \bullet b \bullet b \bullet b \bullet c \bullet c}$.

Look at all the pairs in the radicand.

$\sqrt{(6 \bullet 6) \bullet (a \bullet a) \bullet a \bullet (b \bullet b) \bullet (b \bullet b) \bullet (c \bullet c)}$

Place one number or variable outside the radical for each pair in the radicand, leaving only those numbers or variables in the radicand that are not pairs.

$\sqrt{36a^3b^4c^2}$

$= 6 \bullet a \bullet b \bullet b \bullet c\sqrt{a}$

$= 6ab^2c\sqrt{a}$

Example 3: Simplify $\sqrt{27c^3d^5}$

$= \sqrt{3\bullet3\bullet3\bullet c\bullet c\bullet c\bullet d\bullet d\bullet d\bullet d\bullet d}$

$= \sqrt{(3\bullet3)\bullet3\bullet(c\bullet c)\bullet c\bullet(d\bullet d)\bullet(d\bullet d)\bullet d}$

$= 3\bullet c\bullet d\bullet d\sqrt{3\bullet c\bullet d} = 3cd^2\sqrt{3cd}$

Simplify the following radial expressions showing all your work:

1. $\sqrt{25x^2}$

2. $\sqrt{121a^2b^2}$

3. $\sqrt{40x^3y^4}$

4. $\sqrt{50a^3b^5}$

5. $\sqrt{200xy^2z^3}$

6. $\sqrt{24c^3d^2e}$

7. $\sqrt{60r^3s^2}t^4$

Radical Expressions
Assessment 2

Simplify the following radical expressions showing your work.

1. $\sqrt{16x^2}$

2. $\sqrt{144a^4b^2}$

3. $\sqrt{50c^2d^3}$

4. $\sqrt{24x^3yz^5}$

Radical Expressions
Session 3

We can determine to which number set a solution belongs. All real numbers are either rational or irrational. You have worked with natural numbers, whole numbers, rational numbers, irrational numbers, and integers in previous grades. Let's review these number sets.

The set of **natural numbers** is the set of counting numbers and can be represented by N. **N = {1, 2, 3, 4, 5, 6, …}.**

The set of **whole numbers** includes the set of natural numbers and 0. It can be represented by W. **W = {0, 1, 2, 3, 4, …}.**

The set of **integers** is the set of real numbers consisting of the whole numbers and their opposites. It can be represented by Z. **Z = {… -2, -1, 0, 1, 2, ….}.**

The set of **rational numbers** is the set of real numbers that can be expressed as a quotient of two integers. We can represent the set **of rational numbers by Q.**

The set of **irrational numbers** is the set of real numbers that cannot be expressed as a ratio of two integers. We can represent the set of **irrational numbers by I.**

The set of **real numbers** is the set of rational and irrational numbers. We can represent the set of **real numbers by R**.

Review:

Using a Venn diagram or other graphic organizer, show the relationship between the different number sets to include natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.

Sometimes equations have terms that belong to a number system(s) but the solution belongs to a different number system(s). We can look at equations to try and predict to which number system(s) the solution will belong.

Example 1: The square dog run (fenced area for housing a dog) has a perimeter of 24 ft. Miriam wants to split the dog run diagonally so that she can separate her two dogs. What length of fence will she need to install?

Solution:

24 is a rational, whole, integer, and natural number. The side length of the square is 6 since there are four equal side lengths in a square and the perimeter (24) divided by 4 is 6. When you draw a line diagonally across a square you get two right triangles. See Figure 1 below. You can use the Pythagorean Theorem to find the length of the fence (f) Miriam needs to split the square diagonally. By using the Pythagorean Theorem, you will take the square root of a number. Therefore, the solution may be irrational if the number you take the square root of is not a perfect square. Using the Pythagorean Theorem you find that $6^2+6^2=f^2$. $36+36= f^2$. $72= f^2$. To find out what f equals you need to use the inverse property. The inverse of raising a term to the exponent 2 is finding the square root. Taking the square root of both sides you get

$$\sqrt{72} = \sqrt{f^2}$$

$$\sqrt{3 \bullet 3 \bullet 2 \bullet 2 \bullet 2} = \sqrt{f \bullet f}$$

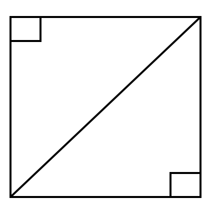
$$\sqrt{(3 \bullet 3) \bullet 2 \bullet (2 \bullet 2)} = \sqrt{(f \bullet f)}$$

$$3 \bullet 2\sqrt{2} = f$$

$$6\sqrt{2} = f$$

The solution is an **irrational number** like we predicted, even though the numbers in the problem were all **rational numbers**. The reason the number system changed was that we had to take the **square root** of a number. Miriam will have to estimate the fence she needs is about 7 and a half feet long.

Figure 1



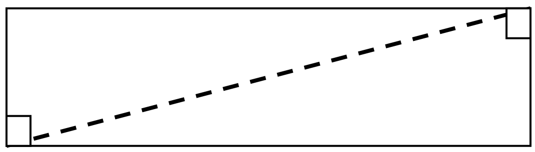
Example 2: Riley was asked to calculate the batting average for Chris, the best hitter on the baseball team. Chris was at bat 17 times this year. He had 6 hits. Will the number system(s) of the solution be the same as the number system(s) of the terms in the problem? Explain your reasoning and prove you are correct.

Solution:

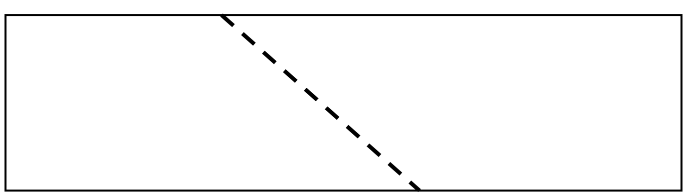
6 and 17 are **rational, integer, whole, and natural numbers**. To calculate batting average the number of "hits" is divided by the number of "at bats". 17 will not divide evenly into 6. Therefore, the solution will not be a whole, natural, or integer. Since a **ratio** can be made with 6 out of 17, the solution will be rational. 6 divided by 17 is .29411764705…Therefore, the solution is a **rational number**.

Predict whether or not the solutions to the problems below will belong to the same number system(s) as the terms in the original problem. Justify your reasoning and prove whether or not your prediction was correct.

1. A rectangular rug with a width of 4 feet and a length of 6 feet is being cut diagonally to fit in a corner of a room. What will the length of the diagonal be?



2. A rectangular table 4 feet wide and 16 feet long is being cut into two trapezoidal tables. In order to cut the tables to fit the room correctly, the length is cut diagonally so that the small base is 6. What would the length of the larger base be? What would the length of the cut be? (Hint: make the trapezoid into a triangle and a rectangle. Use a theorem to find the length of the cut.)



3. Reyna is making a budget and needs to calculate the average electric bill for her apartment. The table below includes the cost of electricity each month this year. What is her average monthly electric bill?

Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
85.12	88.13	77.56	71.00	79.25	94.91	101.00	98.89	98.80	89.25	84.22	86.78

4. Gene, an eleven-year-old, asked his teacher how old she was. Instead of answering, the teacher gave Gene a logic problem that can reveal the teacher’s age. How old is his teacher?

I am more than twice your age but less than three times your age.

When you were beginning winter break, I was one fortieth of a millennium.

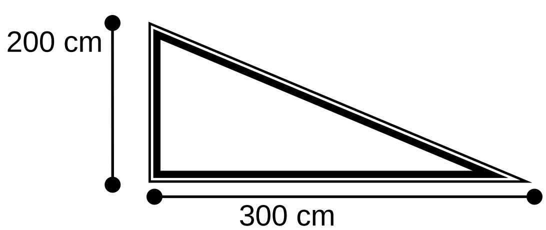
I am now just over a quarter of a century.

Radical Expressions
Assessment 3

Predict whether or not the solution to each problem will belong to the same number system(s) as the terms in the original problem. Justify your prediction. Solve the problem and prove whether or not the prediction was correct.

1. Giovana is balancing her bank account. She withdrew money from the ATM seven times this pay period. She withdrew $20.00 three times, $40.00 twice, $80.00 once, and $140.00 once. She used her own bank's ATM three times and did not have to pay any fees for those withdrawls. She used other banks four times and paid the following fees: $2.50, $3.00, $2.50, and $1.50. She had a beginning balance of $367.50. What is her final balance?

2. The physics class built a ramp. The class is measuring the time it takes for toy cars to roll down the ramp. The ramp is 200 centimeters tall at the highest point and has a horizontal length of 300 centimeters. What is the slope of the ramp?



Radical Expressions
Session 4

Many problems involve radicals. Study the examples. Then solve the problems that follow showing your work. Leave your answer in simplest radical form.

Example 1: The measure of the area of a square is 132 square centimeters. Find the length of a side of this square, rounded to the nearest tenth.

Solution:

Recall that the formula for the area of a square is $A = s^2$.

Since the area is given, substitute that number for A in the formula. $132 = s^2$

To find the measure of the side of the square, we must solve for s. Recall that the square root is the inverse of an exponent of 2. Therefore, $\sqrt{s^2} = \sqrt{132}$.

132 = 2 • 2 • 33 and $s^2 = s \bullet s$

Therefore $\sqrt{s \bullet s}$ = $\sqrt{2 \bullet 2 \bullet 33}$

s = $2\sqrt{33}$

$\sqrt{33}$ is between the perfect squares $\sqrt{25}$ & $\sqrt{36}$. That means it is between 5 and 6, and is closer to 6. A good estimate would be 5.7 because it is almost ¾ of the way between 5 and 6.

The solution can be estimated as 2•5.7=11.4.

The side is approximately 11.4 cm.

Example 2: If two sides of a right triangle are 4 and 6, find the hypotenuse.

Remember that the **<u>Pythagorean Theorem</u>** tells us how to find the hypotenuse of a right triangle given the two sides of the right triangle. If c is the hypotenuse and a and b are the sides of the right triangle, then $c^2 = a^2 + b^2$

Solution:

$$c^2 = a^2 + b^2$$

$$c^2 = 4^2 + 6^2$$

$$c^2 = 52$$

$$c^2 = 2 \bullet 2 \bullet 13$$

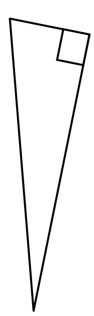
$$\sqrt{c^2} = \sqrt{2 \bullet 2 \bullet 13}$$

$$\sqrt{(c \bullet c)} = \sqrt{(2 \bullet 2) \bullet 13}$$

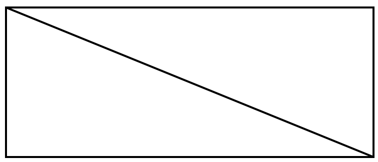
c = $2\sqrt{13}$

Find the solutions to the following problems. It may be helpful to label the diagram first. Show your work and explain your answer. Leave your answers in simplest radical form.

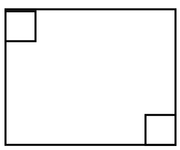
1. If two sides of a right triangle are 3 and 15, find the hypotenuse.



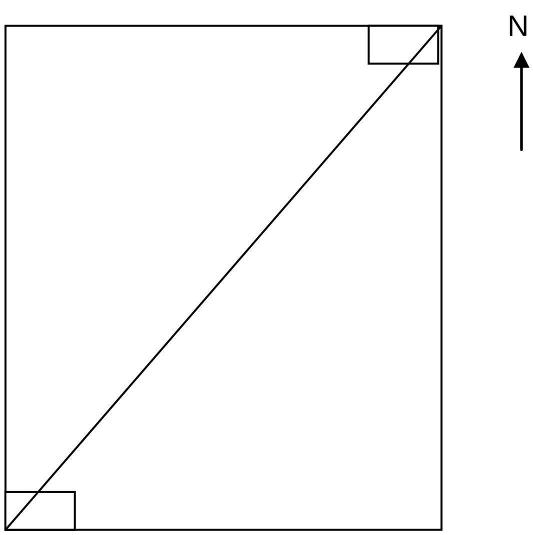
2. The diagonal of a rectangle is 12 cm and the width is 7 cm. Find the area of the rectangle.



3. The measure of the area of a square is 98 square centimeters. Find the length of a side of this square.



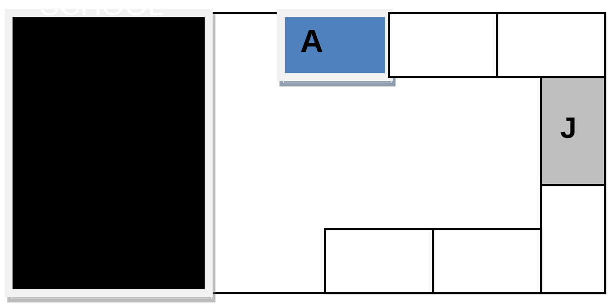
4. Emily is walking home from school. She decides to cut across a rectangular park by walking on a diagonal path that cuts through the park. If the north side of the park is 30 feet and the west side is 40 feet, how long is the path? How much less does Emily walk by using the diagonal path than by walking along the sides of the park?



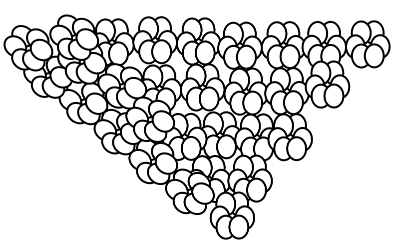
Radical Expressions
Assessment 4

Solve the following problems showing your work. It will be helpful to label the diagram. Leave answers in simplest radical form unless otherwise specified. Make sure to answer all questions.

1. Anthony is going to Jason's, house for dinner. Rather than walking the two blocks to his friend's house, Anthony will cut a diagonal path through the school playground. The distance from Anthony's house to Jason's street is 84 feet. The distance from that intersection to Jason's house is 60 feet. How far will Anthony travel along the diagonal path (round your answer to the nearest tenth)? How many feet will he save by taking the shortcut rather than walking along the sidewalks to Jason's house?



2. Maria is planting a flower garden. Her garden will be shaped like a right triangle. She is going to plant different types of flowers along each edge. The two shorter edges are 15 feet and 15 feet. How long is the other edge? What is the total amount of fencing that Maria needs to buy to place a fence along each edge?



Extensions

1. Find the product of two radicals and simplify their answer.
2. Find the cube roots of algebraic expressions.

Radical Expressions
Extension

To find the product of two radical expressions, multiply what is outside the radical of one radical by what is outside the other radical. Multiply the radicand of one radical by the radicand of the other radical. Then simplify your answer if needed.

The radical is a special grouping symbol. You can combine radicands that are being multiplied, but you cannot combine radicands with terms outside of the radical.

Example 1:

$4\sqrt{3} \bullet \sqrt{7}$ can be simplified to $4\sqrt{3 \bullet 7}$

$= 4\sqrt{21}$

Example 2:

Find $3\sqrt{2} \bullet 4\sqrt{3}$

$3\sqrt{2} \bullet 4\sqrt{3}$

$= 3 \bullet 4\sqrt{2 \bullet 3}$

$= 12\sqrt{6}$

Example 3:

Find $5\sqrt{10} \bullet 7\sqrt{10}$

$= 5 \bullet 7\sqrt{10 \bullet 10}$

$= 35 \bullet 10 = 350$

Example 4:

Find $(-3\sqrt{3})(2\sqrt{15})$

$= -3 \bullet 2\sqrt{3 \bullet 15}$

$= -6\sqrt{3 \bullet 3 \bullet 5}$

$= -6 \bullet 3\sqrt{5}$

$= -18\sqrt{5}$

Example 5:

Find $(2\sqrt{5})(7\sqrt{30})$

$= 2 \bullet 7\sqrt{5 \bullet 30}$

$= 14\sqrt{5 \bullet 2 \bullet 3 \bullet 5}$

$= 14 \bullet 5\sqrt{2 \bullet 3}$

$= 70\sqrt{6}$

Find the product of each of the following set of radical expressions showing your work.

1. $(3\sqrt{5})(7\sqrt{11})$

2. $2\sqrt{6} \bullet 4\sqrt{6}$

3. $(-5\sqrt{5})(8\sqrt{10})$

4. $(12\sqrt{3})(-2\sqrt{27})$

5. $11\sqrt{7} \bullet 3\sqrt{14}$

6. $(-10\sqrt{5})(-4\sqrt{50})$

We can find the cube root of a radical expression in the same way as we found the square root of a radical expression. Instead of finding a pair of numbers under the radical, we find a group of three numbers or variables that are the same. Then we can take each group out and it becomes one of those numbers.

Example 1:

Simplify $\sqrt[3]{a^3b^3}$

$$\sqrt[3]{a^3b^3} = \sqrt[3]{a \bullet a \bullet a \bullet b \bullet b \bullet b}$$

Because the index is 3 (we are trying to find the cube root), we look for groups of three numbers or variables under the radical.

$$\sqrt[3]{a \bullet a \bullet a \bullet b \bullet b \bullet b}$$

$$= \sqrt[3]{(a \bullet a \bullet a) \bullet (b \bullet b \bullet b)}$$

$$= ab$$

Example 2:

Simplify $\sqrt[3]{x^4y^5}$

$$\sqrt[3]{x^4y^5} =$$

$$\sqrt[3]{x \bullet x \bullet x \bullet x \bullet y \bullet y \bullet y \bullet y \bullet y}$$

$$= \sqrt[3]{(x \bullet x \bullet x) \bullet x \bullet (y \bullet y \bullet y) \bullet y \bullet y}$$

$$= x \bullet y \quad \sqrt[3]{x \bullet y \bullet y}$$

$$= xy \ \sqrt[3]{xy^2}$$

(Remember to write the index in your answer.)

Example 3:

Simplify $\sqrt[3]{27a^4}$

$$\sqrt[3]{27a^4} = \sqrt[3]{3 \bullet 3 \bullet 3 \bullet a \bullet a \bullet a \bullet a}$$

$$= \sqrt[3]{3 \bullet 3 \bullet 3 \bullet a \bullet a \bullet a \bullet a}$$

$$= \sqrt[3]{(3 \bullet 3 \bullet 3) \bullet (a \bullet a \bullet a) \bullet a}$$

$$= 3 \bullet a \bullet \sqrt[3]{a}$$

$$= 3a \sqrt[3]{a}$$

Example 4:

Simplify $7 \sqrt[3]{200m^5n}$

$$7 \sqrt[3]{200m^5n} = 7 \bullet \sqrt[3]{2 \bullet 10 \bullet 10 \bullet m \bullet m \bullet m \bullet m \bullet m \bullet n}$$

$$= 7 \bullet \sqrt[3]{2 \bullet 2 \bullet 2 \bullet 5 \bullet 5 \bullet m \bullet m \bullet m \bullet m \bullet m \bullet n}$$

$$= 7 \bullet \sqrt[3]{(2 \bullet 2 \bullet 2) \bullet 5 \bullet 5 \bullet (m \bullet m \bullet m) \bullet m \bullet m \bullet n}$$

$$= 7 \bullet 2 \bullet m \bullet \sqrt[3]{5 \bullet 5 \bullet m \bullet m \bullet n}$$

$$= 14m \sqrt[3]{25m^2n}$$

Simplify the following radical expressions showing your work.

1. $\sqrt[3]{64}$

2. $\sqrt[3]{27x^3y^3}$

3. $\sqrt[3]{16a^4}$

4. $\sqrt[3]{24s^7}$

5. $\sqrt[3]{c^2d^3e^4}$

6. $\sqrt[3]{81x^3yz^5}$

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards, p. 292-294
2008 The Final Report of the National Mathematics Advisory Panel, p. 16