Distance and Midpoint
An ADE Mathematics Lesson
Days 6-10

Author	ADE Content Specialists
Grade Level	9th grade
Duration	Five days

Aligns To

Mathematics HS:
Strand 1: Number and Operations
Concept 1: Number Sense
PO 3. Express that the distance between two numbers is the absolute value of their difference.
Concept 2: Numerical Operations
PO 1. Solve word problems involving absolute value, powers, roots, and scientific notation.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO 1. Determine how to find the midpoint between two points in the coordinate plane.
PO 2. Illustrate the connection between the distance formula and the Pythagorean Theorem.
PO 3. Determine the distance between two points in the coordinate plane.

Strand 5: Structure and Logic
Concept 1: Algorithms and Algorithmic Thinking
PO 1. Select an algorithm that explains a particular mathematical process; determine the purpose of a simple mathematical algorithm.
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 1. Analyze a problem situation, determine the question(s) to be answered, organize given information, determine how to represent the problem, and identify implicit and explicit assumptions that have been made.
PO 2. Solve problems by formulating one or more strategies, applying the strategies, verifying the solution(s), and communicating the reasoning used to obtain the solution(s).

Connects To

Mathematics HS:
Strand 1: Number and Operations
Concept 3: Estimation
PO 2. Use estimation to determine the reasonableness of a solution.

Strand 4: Geometry and Measurement
Concept 3: Coordinate Geometry
PO4. Verify characteristics of a given geometric figure using coordinate formulas for distance, midpoint, and slope to confirm parallelism, perpendicularity, and congruence.

Strand 5: Structure and Logic
Concept 2: Logic, Reasoning, Problem Solving, and Proof
PO 3. Evaluate a solution for reasonableness and interpret the meaning of the solution in the context of the original problem.

Overview
In this lesson, midpoint and distance are explored algebraically and geometrically. You also solve contextual problems involving midpoint and distance. You will study the connection between the distance formula and the Pythagorean Theorem.

Purpose
It is often necessary to solve problems that involve finding either midpoints or distances. It is important to understand exactly what finding the midpoint and distance means. Understanding distance helps you to understand absolute value.

Materials
- Midpoint and distance worksheets
- Ruler
- Graph paper

Objectives
Students will:
- Determine how to find the midpoint between two points.
- Determine how to find the distance on a number line between two values or between two points on a coordinate graph.
- Understand the connection between the Pythagorean Theorem and the Distance Formula.
- Be able to solve contextual problems involving distance or midpoint.

Lesson Components

Prerequisite Skills: This lesson builds upon previous skills of finding rational and irrational points on a number line. Other important skills are applying the meaning of absolute value and graphing points on a coordinate plane.

Vocabulary: *distance, midpoint, Pythagorean Theorem, Midpoint Formula, Distance Formula*

Session 1: Midpoint (1 day)
1. Determine both formally and informally how to find the midpoint between two points. Problems include finding the midpoint given both endpoints, and finding the unknown endpoint given the midpoint and one endpoint.

Session 2: Distance (4 days)
1. Illustrate the connection between the distance formula and the Pythagorean Theorem.
2. Determine the distance between two points in the coordinate plane.

Assessment
There are assessments embedded after each session that pinpoint misconceptions about specific topics.

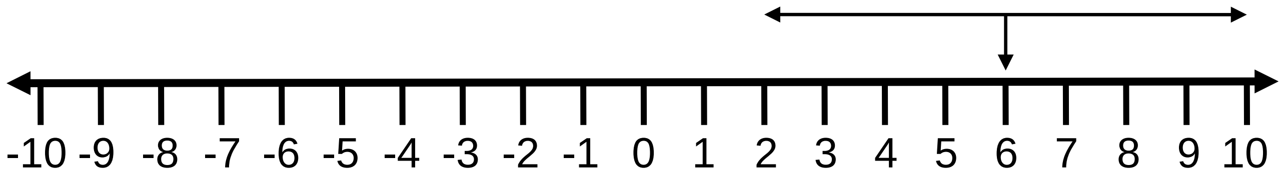
Distance and Midpoint
Session 1 - Midpoint

We can find the midpoint on a number line or the midpoint between two points on a coordinate plane.

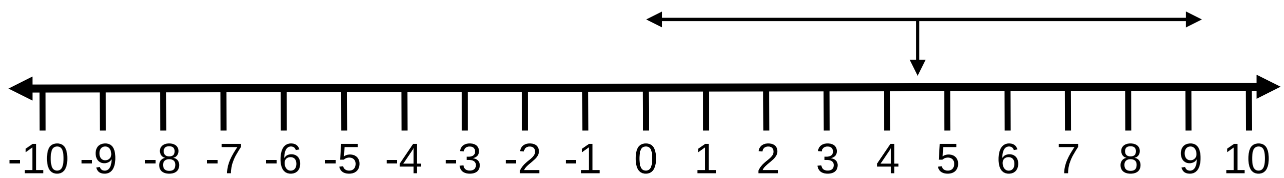
To find the **midpoint** between two values on a number line, find the value that is half-way between the two numbers.

Look at the placements of the numbers on the number line to solve the following problems. The midpoint is the number that is the same distance or equidistant from both numbers on the number line.

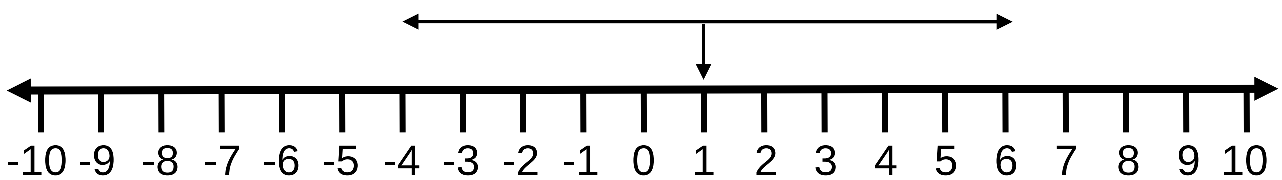
Example 1: Find the midpoint between 2 and 10. If we count from each side the same number of units, we find that the midpoint between 2 and 10 is 6.



Example 2: Find the midpoint between 0 and 9. If we count from each side the same number of units, we find that the midpoint between 0 and 9 is 4.5 or $4\frac{1}{2}$.



Example 3: Find the midpoint between -4 and 6. If we count from the same number of units from each side, we find that the midpoint between -4 and 6 is 1.



We also find the **midpoint** between two numbers by adding the numbers together and dividing the sum by 2.

Let a = the first value. Let b = the second value. Let m = the midpoint.

$$m = \frac{a+b}{2}$$

Do we get the same answer for the examples above if we use this formula?

Example 1: Find the midpoint between 2 and 10.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{2+10}{2} = \frac{12}{2} = 6$$

Example 2: Find the midpoint between 0 and 9.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{0+9}{2} = \frac{9}{2} = 4.5$$

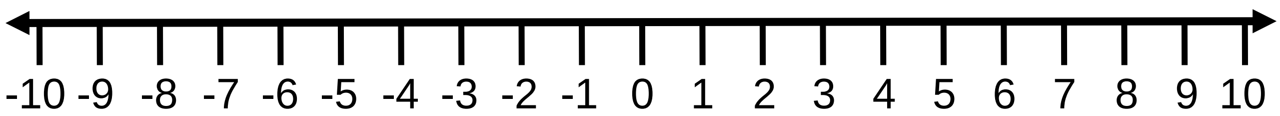
Example 3: Find the midpoint between -4 and 6.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{-4+6}{2} = \frac{2}{2} = 1$$

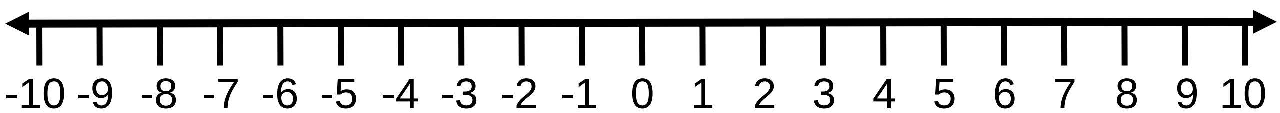
In each example, we arrived at the same midpoint whether we used the number line or the formula to find the midpoint. What is important is that you understand what finding the midpoint between two values means.

Find the midpoint in the following problems by using a number line.

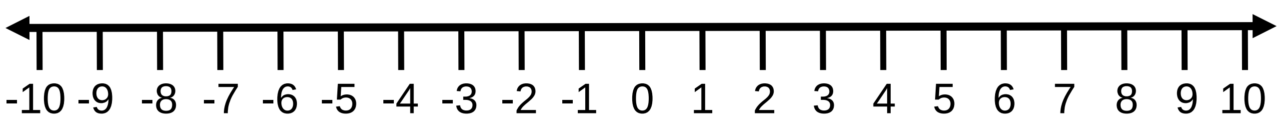
1. Find the midpoint between 1 and 9.



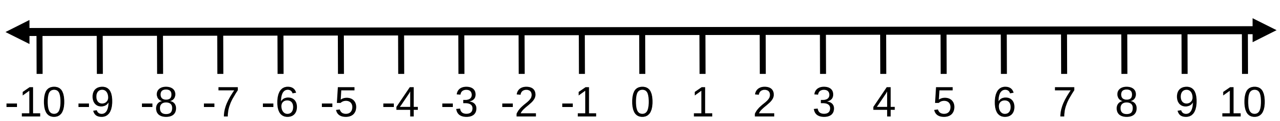
2. Find the midpoint between 0 and 5.



3. Find the midpoint between -6 and 2.



4. Find the midpoint between -8 and -1.



Find the midpoint by using the midpoint formula: $m = \frac{a+b}{2}$. Show all your work.

1. Find the midpoint between 20 and 6.

2. Find the midpoint between 12 and 40.

3. Find the midpoint between -10 and 100.

4. Find the midpoint between -20 and 5.

5. Find the midpoint between -30 and -15.

Sometimes, we are given one endpoint and the midpoint and asked to find the other endpoint.

We still use the formula, $m = \frac{a+b}{2}$, and substitute the values in for *m* and *a* or *b*. It does not matter whether we substitute the given endpoint for *a* or *b* as we will obtain the same answer.

Example 1:

The midpoint of two endpoints is 4. If one endpoint is 6, what is the other endpoint?

Solution: Let *m* = 4 and *b* = 6.

$$m = \frac{a+b}{2}$$

$$4 = \frac{a+6}{2}$$

$$4 \bullet 2 = a + 6$$

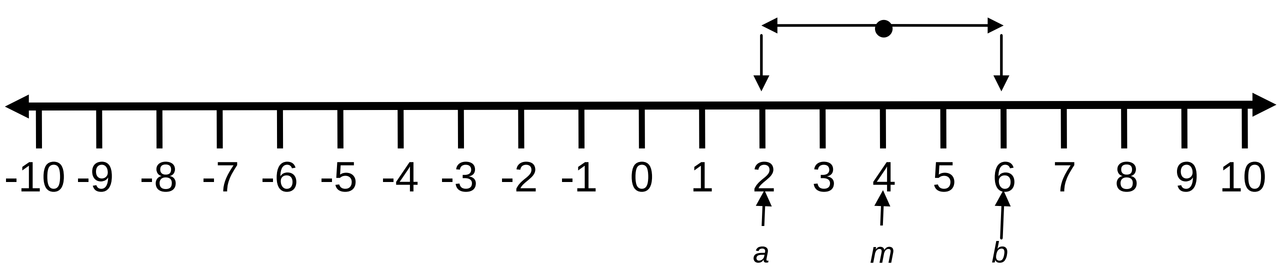
$$8 = a + 6$$

$$a = 2$$

We can check our answer by substituting the value of the endpoints in the midpoint formula to see if we obtain the correct midpoint.

$$m = \frac{a+b}{2} \Rightarrow m = \frac{2+6}{2} = \frac{8}{2} = 4 \text{ (given midpoint)}$$

Note the representation of this problem on the number line.



Note that *b* is 6. This is 2 units to the right of *m,* which is 4. So *a* must be 2 units to the left of 4, which is 2.

Example 2:

The midpoint of two endpoints is -1. If one endpoint is -7, what is the other endpoint?

Solution:

Let *a* = -7 and *m* = -1.

$$m = \frac{a+b}{2}$$

$$-1 = \frac{-7+b}{2}$$

$$-1(2) = -7 + b$$

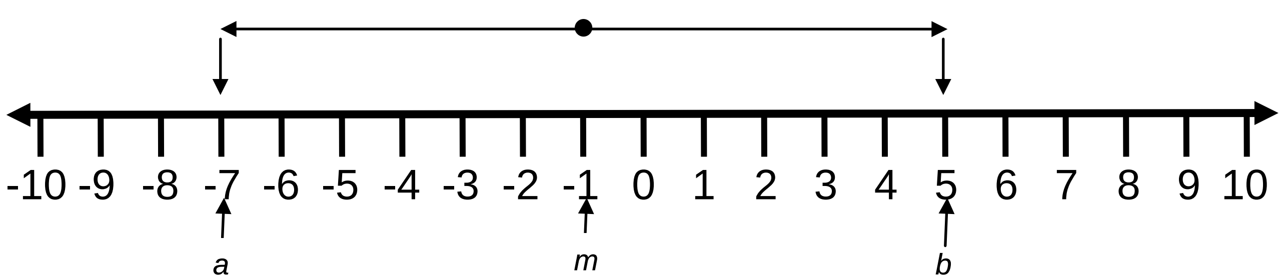
$$-2 = -7 + b$$

$$b = 5$$

We can check our answer by substituting the value of the endpoints in the midpoint formula to see if we obtain the correct midpoint.

$$m = \frac{a+b}{2} \Rightarrow \frac{-7+5}{2} = \frac{-2}{2} = -1 \text{ (given midpoint)}$$

Note the representation of this problem on the number line.



Note that *a* is -7. This is 6 units to the left of *m,* which is -1. So *b* must be 6 units to the right of -1, which is 5.

Solve the following problems showing all your work:

1. The midpoint of two endpoints is 7. If one endpoint is 4, what is the other endpoint?

   Remember that $m = \frac{a+b}{2}$.

2. The midpoint of two endpoints is 2. If one endpoint is -5, what is the other endpoint?

Distance and Midpoint
Assessment 1

Find the midpoint of the following pairs of endpoints using the both a number line and the midpoint formula. Show all your work.

1. 2 and 6

2. 4 and 10

3. -3 and -8

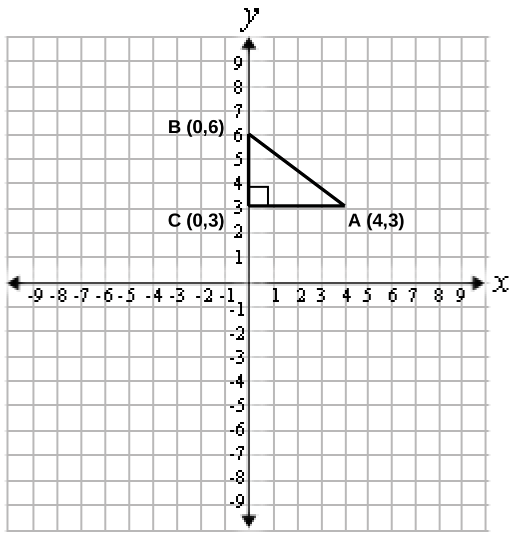
4. -10 and 4

Distance and Midpoint
Session 2 – Distance

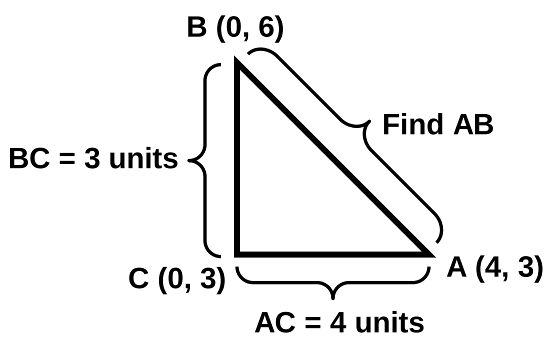
There are different ways to find the distance between two points in the coordinate plane.

Method 1: We can <u>use the Pythagorean Theorem</u> to find the distance between two points.

Example 1: Find the distance between points A and B on the coordinate grid below.



Solution: The vertices of the triangle are A (4, 3), B (0, 6) and C (0, 3). If we count on the coordinate grid, we find the length of AC is 4 units and the length of BC is 3 units. We have a right triangle so we can use the Pythagorean Theorem to find the length of AB.



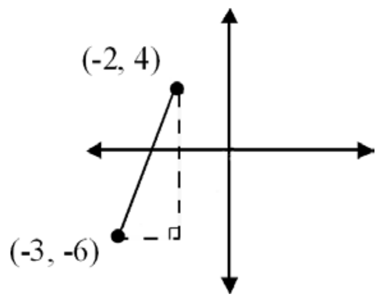
Use the Pythagorean Theorem $c^2 = a^2 + b^2$. Remember that *a* and *b* are the legs of the right triangle and *c* is the hypotenuse. Each side is opposite each angle. Side *a* is 3 units, side *b* is 4 units. Find side *c*.

$$c^2 = a^2 + b^2 \Rightarrow c^2 = 3^2 + 4^2 \Rightarrow c^2 = 9 + 16 = 25 \Rightarrow c = \sqrt{25} = 5.$$

Therefore the length of AB is 5 and the distance between points A and B is 5 units.

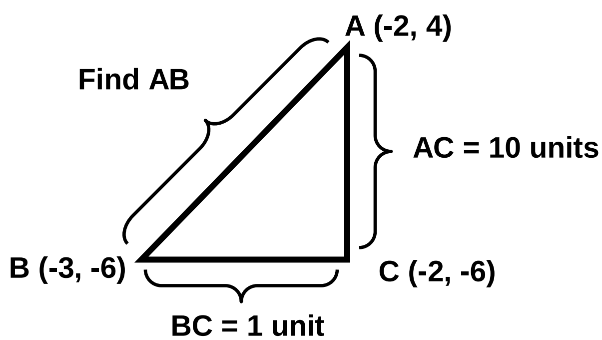
Example 2:

Find the distance between the given points (-2, 4) and (-3, -6) in the diagram given below.



Solution:

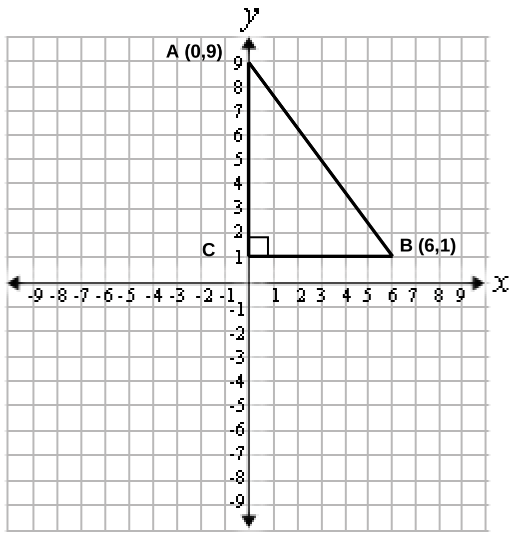
We must first determine the coordinates of the missing point which is the vertex of the right angle. Since this is a right angle, we know that the x-value of the point is the same as (-2, 4) so the x-value is -2. The y-value will be the same as in the point (-3, -6) so the y-value is -6. Therefore the value of the missing point is (-2, -6). Let's redraw this triangle to find the distance between the given points.



Use the Pythagorean Theorem $c^2 = a^2 + b^2 \Rightarrow c^2 = 1^2 + 10^2 = 101 \Rightarrow c = \sqrt{101}$. Therefore the distance between points A and B is $\sqrt{101}$. This is expressed in simplest form. We can estimate this difference by using a calculator to find an approximate value, $\sqrt{101} \approx$ 10.05 units.

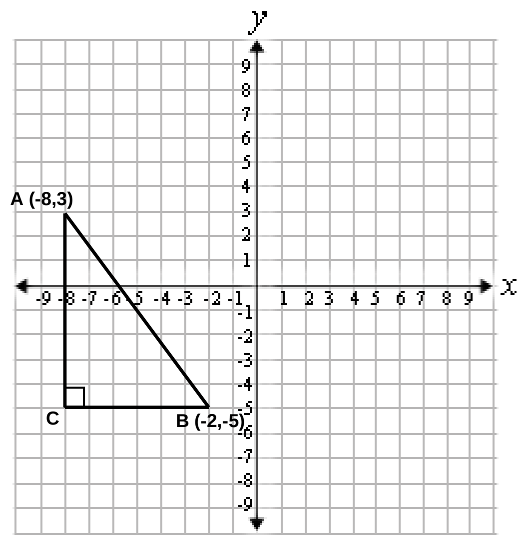
Problem 1:

Find the distance between the given points in the diagram below using the Pythagorean Theorem. Show all your work in the space provided that follows the diagram.



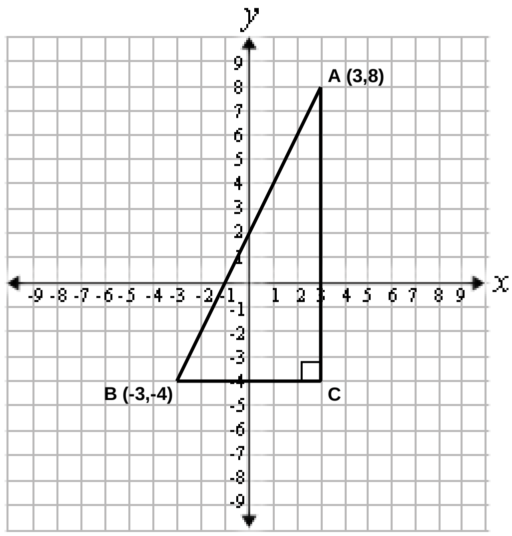
Problem 2:

Find the distance between the given points in the diagram below using the Pythagorean Theorem. Show all your work in the space provided that follows the diagram.

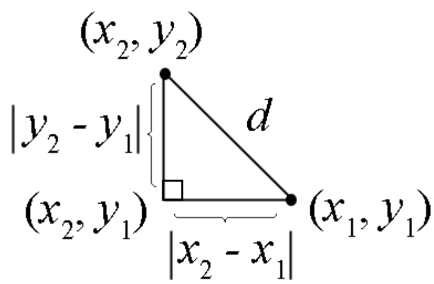


Problem 3:

Find the distance between the given points in the diagram below using the Pythagorean Theorem. Show all your work in the space provided that follows the diagram.



Think about the examples shown previously and the problems that you worked. Then, study the diagram below.



Each coordinate point is labeled. Just as we found the length of each side in each previous triangle, the length of each side is given in this diagram. We are then left to find *d*.

Using the Pythagorean Theorem, we determine that $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ given the vertices of the triangle. In this diagram, we have replaced side "a" by $|x_2 - x_1|$ and side "b" by $|y_2 - y_1|$, and side "c" by *d*.

Then by using the Pythagorean Theorem, we know that $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

Taking the square root of each side gives us the <u>distance formula</u>

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **.** Since we are squaring each quantity, the result will be positive and there is no longer a need for the absolute value sign.

The <u>distance formula</u> is a second method for finding the distance between two points on a coordinate grid. Given points (x_1, y_1) and (x_2, y_2), the distance between these points is determined by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Method 2: We can find the distance between the given points, (x_1, y_1) and (x_2, y_2) by using the <u>distance formula</u>.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let's examine one of our previous examples.

Example 1:

Find the distance between the points (4, 3) and (0, 6).

Solution:

(x_1, y_1) = (4, 3) and (x_2, y_2) = (0, 6). Therefore, $x_1 = 4$, $x_2 = 0$, $y_1 = 3$ and $y_2 = 6$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow d = \sqrt{(0-4)^2 + (6-3)^2}$$

$$d = \sqrt{(-4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$d = 5$$

This is the same answer that we obtained by using the coordinate graph in our previous example 1 under Method 1.

Note – It does not matter which point is designated as (x_1, y_1) and which point is designated by (x_2, y_2), the result will be exactly the same. Try the above example designating the first point as (x_1, y_1) and the second point as (x_2, y_2).

(x_1, y_1) = (0, 6) and (x_2, y_2) = (4, 3). Therefore, $x_1 = 0$, $x_2 = 4$, $y_1 = 6$ and $y_2 = 3$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow d = \sqrt{(4-0)^2 + (3-6)^2}$$

$$d = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25}$$

$$d = 5$$

This is the same answer that we got when we labeled the two points differently.

Let's clarify our thinking before moving on to more examples using the second method to find the distance between two points on a coordinate plane.

Think about the information, examples, and problems that we have worked and then answer the following questions.

1. Explain the relationship between a right triangle and finding the distance between two points on a coordinate plane.

2. Explain how to find the distance between two points on a coordinate plane by drawing a right triangle and using the Pythagorean Theorem.

3. What is the formula to find the distance between two points on a coordinate grid? Explain briefly how that formula is derived in your own words.

4. Does it matter in which order the points are put into the distance formula? Explain your answer.

Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, find the distance between the given points in the earlier three problems (pages 13-15) you found using the Pythagorean Theorem.

1. Find the distance between the points (0, 9) and (6, 1). Show all your work in the space provided.

2. Find the distance between the points (-8, 3) and (-2, -5). Show all your work in the space provided.

3. Find the distance between the points (3, 8) and (-3, -4). Show all your work in the space provided.

Using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, find the distance between the given points in the space provided.

4. Find the distance between the points (2, 4) and (8, -4). Show all your work in the space provided.

5. Find the distance between the points (-3 0) and (-6, -4). Show all your work in the space provided.

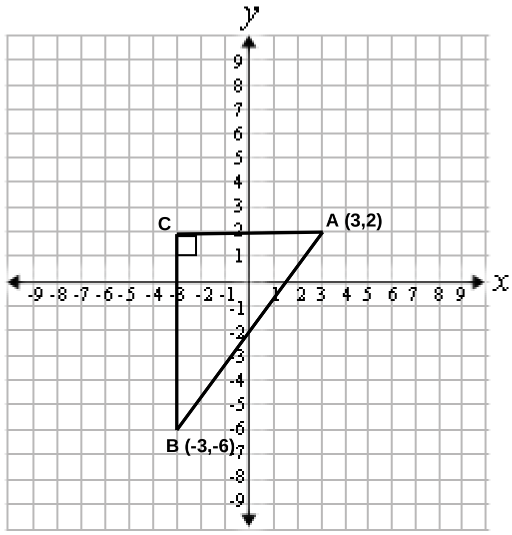
6. Find the distance between the points (10, 7) and (-2, 2). Show all your work in the space provided.

In the space provided below, show that it does not make a difference in what order you place the points in the distance formula. Show all of your work.

- Choose any two points in the coordinate plane.
- Use the distance formula twice.
- The first time assign (x_1, y_1) to the first point you list.
- The second time assign (x_1, y_1) to the second point you list.
- Show that your answers are the same.

Distance and Midpoint
Assessment 2

Find the length of AB on the coordinate plane by using the Pythagorean Theorem. Show all your work in the space provided following the graph.



Distance and Midpoint
Assessment 3

Find the distance between the following set of points using the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Show all your work in the space provided.

1. Find the distance between the points (4, 6) and (-2, -2).

2. Find the distance between the points (9, 5) and (-1, 2).

Distance and Midpoint
Assessment 4

1. Answer the following questions true or false. Justify your answer.

    a. Distance is always positive.

    b. The distance formula that enables us to find the distance between two points on a coordinate plane can be derived from the Pythagorean Theorem.

    c. When trying to use the distance formula, it makes a difference in the correctness of the answer if the first point is not labeled (x_1, y_1) and the second point is not labeled (x_2, y_2).

2. Complete the sentence with the correct phrase(s). Explain your answer.

    a. To find the midpoint of a line segment, ______________________.

    b. To find the distance between two points on a coordinate plane, ____.

3. Explain the relationship between the number line and the midpoint formula to find the midpoint between two sets of points.

Extensions

1. In the extension lesson, you will simplify radicals after applying the distance formula.

Distance and Midpoint
Extension

To find the distance between two points, (x_1, y_1) and (x_2, y_2), use the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Distance is always positive.

It is sometimes necessary to simplify the radical after using the distance formula. Study the following examples.

Example 1:

Find the distance between the points (1, -3) and (-5, 0) using the distance formula.

Solution:

Let (x_1, y_1) = (1, -3) and (x_2, y_2) = (-5, 0). Then, $x_1 = 1$, $x_2 = -5$, $y_1 = -3$, and $y_2 = 0$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow d = \sqrt{(-5-1)^2 + (0-(-3))^2}$$

$$d = \sqrt{(-6)^2 + 3^2} = \sqrt{36+9} = \sqrt{45}$$

$$\sqrt{45} = \sqrt{9 \bullet 5} = 3\sqrt{5}$$

The distance between (1, -3) and (-5, 0) is $3\sqrt{5}$.

Example 2:

Find the distance between the points (6, 4) and (2, -4) using the distance formula.

Solution:

Let (x_1, y_1) = (2, -4) and (x_2, y_2) = (6, 4). Then, $x_1 = 2$, $x_2 = 6$, $y_1 = -4$, and $y_2 = 4$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \;\Rightarrow\; d = \sqrt{(6-2)^2 + (4-(-4))^2}$$

$$d = \sqrt{4^2 + 8^2} = \sqrt{16+64} = \sqrt{80}$$

$$\sqrt{80} = \sqrt{16 \bullet 5} = 4\sqrt{5}$$

The distance between (6, 4) and (2, -4) is $4\sqrt{5}$.

Example 3:

Find the distance between the points (5, 2) and (-2, -5) using the distance formula.

Solution:

Let (x_1, y_1) = (5, 2) and (x_2, y_2) = (-2, -5). Then, $x_1 = 5$, $x_2 = -2$, $y_1 = 2$, and $y_2 = -5$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \;\Rightarrow\; d = \sqrt{(-2-5)^2 + (-5-2)^2}$$

$$d = \sqrt{(-7)^2 + (-7)^2} = \sqrt{49+49} = \sqrt{98}$$

$$\sqrt{98} = \sqrt{49 \bullet 2} = 7\sqrt{2}$$

The distance between the points (5, 2) and (-2, -5) is $7\sqrt{2}$.

Solve the problems below using the distance formula showing all your work in the space provided. Place your answer in simplest radical form.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. Find the distance between the points (4, 0) and (-6, 2).

2. Find the distance between the points (-5, 3) and (5, -2).

3. Find the distance between the points (8, -2) and (4, 10).

Sources

2008 AZ Mathematics Standards
2000 NCTM Principles and Standards, p. 290-294 & 308-318
2008 The Final Report of the National Mathematics Advisory Panel, p. 16 & p.28